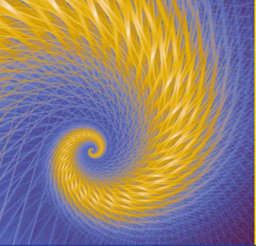

Jacques Janssen
Raimondo Manca

Semi-Markov Risk Models for Finance, Insurance and Reliability

Springer

SEMI-MARKOV RISK MODELS FOR FINANCE, INSURANCE AND RELIABILITY

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PREFACE

This book aims to give a complete and self-contained presentation of semi-Markov models with finitely many states, in view of solving real life problems of risk management in three main fields: Finance, Insurance and Reliability providing a useful complement to our first book (Janssen and Manca (2006)) which gives a theoretical presentation of semi-Markov theory. However, to help assure the book is self-contained, the first three chapters provide a summary of the basic tools on semi-Markov theory that the reader will need to understand our presentation. For more details, we refer the reader to our first book (Janssen and Manca (2006)) whose notations, definitions and results have been used in these four first chapters.

Nowadays, the potential for theoretical models to be used on real-life problems is severely limited if there are no good computer programs to process the relevant data. We therefore systematically propose the basic algorithms so that effective numerical results can be obtained. Another important feature of this book is its presentation of both homogeneous and non-homogeneous models. It is well known that the fundamental structure of many real-life problems is non-homogeneous in time, and the application of homogeneous models to such problems gives, in the best case, only approximated results or, in the worst case, nonsense results.

This book addresses a very large public as it includes undergraduate and graduate students in mathematics and applied mathematics, in economics and business studies, actuaries, financial intermediaries, engineers and operation researchers, but also researchers in universities and rd departments of banking, insurance and industry.

Readers who have mastered the material in this book will see how the classical models in our three fields of application can be extended in a semi-Markov environment to provide better new models, more general and able to solve problems in a more adapted way. They will indeed have a new approach giving a more competitive knowledge related to the complexity of real-life problems.

Let us now give some comments on the contents of the book.

As we start from the fact that the semi-Markov processes are the children of a successful marriage between renewal theory and Markov chains, these two topics are presented in Chapter 2.

The full presentation of Markov renewal theory, Markov random walks and semi-Markov processes, functionals of (J-X) processes and semi-Markov random walks is given in Chapter 3 along with a short presentation of non-homogeneous Markov and semi-Markov processes.

Chapter 4 is devoted to the presentation of discrete time semi-Markov processes, reward processes both in undiscounted and discounted cases, and to their numerical treatment.
Chapter 5 develops the Cox-Ross-Rubinstein or binomial model and semi-Markov extension of the Black and Scholes formula for the fundamental problem of option pricing in finance, including Greek parameters. In this chapter, we must also mention the presence of an option pricing model with arbitrage possibility, thus showing how to deal with a problem stock brokers are confronted with daily.
Chapter 6 presents other general finance and insurance semi-Markov models with the concepts of exchange and dated sums in stochastic homogeneous and non-homogeneous environments, applications in social security and multiple life insurance models.
Chapter 7 is entirely devoted to insurance risk models, one of the major fields of actuarial science; here, too, semi-Markov processes and diffusion processes lead to completely new risk models with great expectations for future applications, particularly in ruin theory.
Chapter 8 presents classical and semi-Markov models for reliability and credit risk, including the construction of rating, a fundamental tool for financial intermediaries.
Finally, Chapter 9 concerns the important present day problem of pension evolution, which is clearly a time non-homogeneous problem. As we need here more than one time variable, we introduce the concept of generalised non-homogeneous semi-Markov processes. A last section develops generalised non homogeneous semi-Markov models for salary line evolution.
Let us point out that whenever we present a semi-Markov model for solving an applied problem, we always summarise, before giving our approach, the classical existing models. Therefore the reader does not have to look elsewhere for supplementary information; furthermore, both approaches can be compared and conclusions reached as to the efficacy of the semi-Markov approach developed in this book.
It is clear that this book can be read by sections in a variety of sequences, depending on the main interest of the reader. For example, if the reader is interested in the new approaches for finance models, he can read the first four chapters and then immediately Chapters 5 and 6, and similarly for other topics in insurance or reliability.
The authors have presented many parts of this book in courses at several universities: Université Libre de Bruxelles, Vrije Universiteit Brussel, Université de Bretagne Occidentale (EURIA), Universités de Paris 1 (La Sorbonne) and Paris VI (ISUP), ENST-Bretagne, Université de Strasbourg, Universities of Roma (La Sapienza), Firenze and Pescara.
Our common experience in the field of solving some real problems in finance, insurance and reliability has joined to create this book, taking into account the remarks of colleagues and students in our various lectures. We hope to convince

potential readers to use some of the proposed models to improve the way of modelling real-life applications.

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Keywords :

Semi-Markov processes, Homogeneous, Non homogeneous, Risk Management, Finance, Insurance, Reliability, Models, Numerical results, Real Applications.

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Chapter 1

PROBABILITY TOOLS FOR STOCHASTIC MODELLING

In this chapter, the reader will find a short summary of the basic probability tools useful for understanding of the following chapters. A more detailed version including proofs can be found in Janssen and Manca (2006). We will focus our attention on stochastic processes in discrete time and continuous time defined by sequences of random variables.

1 THE SAMPLE SPACE

The basic concrete notion in probability theory is that of the *random experiment*, that is to say an experiment for which we cannot predict in advance the outcome. With each random experiment, we can associate the so-called *elementary events* ω, and the set of all these events Ω is called the *sample space*. Some other subsets of Ω will represent possible *events*. Let us consider the following examples.

Example 1.1 If the experiment consists in the measurement of the lifetime of an integrated circuit, then the sample space is the set of all non-negative real numbers $\mathbb{R}^+$. Possible events are $[a,b],(a,b),[a,b),(a,b]$ where for example the event $[a,b)$ means that the lifetime is at least a and strictly inferior to b.

Example 1.2 An insurance company is interested in the number of claims per year for its portfolio. In this case, the sample space is the set of natural numbers $\mathbb{N}$.

Example 1.3 A bank is to invest in some shares; so the bank looks to the history of the value of different shares. In this case, the sample space is the set of all non-negative real numbers $\mathbb{R}^+$.

To be useful, the set of all possible events must have some properties of stability so that we can generate new events such as:

(i) *the complement* A^c : $A^c = \{\omega \in \Omega : \omega \notin A\}$, (1.1)

(ii) *the union* $A \bigcup B$: $A \bigcup B = \{\omega : \omega \in A \text{ or} \omega \in B\}$, (1.2)

(iii) *the intersection* $A \bigcap B$: $A \bigcap B = \{\omega : \omega \in A, \omega \in B\}$. (1.3)

More generally, if $(A_n, n \geq 1)$ represents a sequence of events, we can also consider the following events:

$$\bigcup_{n\geq 1} A_n, \ \bigcap_{n\geq 1} A_n \tag{1.4}$$

representing respectively the *union* and the *intersection* of all the events of the given sequence. The first of these two events occurs iff at least one of these events occurs and the second iff all the events of the given sequence occur. The set Ω is called the *certain event* and the set $\varnothing$ the *empty event*. Two events A and B are said to be *disjoint* or *mutually exclusive* iff

$$A \bigcap B = \varnothing . \tag{1.5}$$

Event A *implies* event B iff

$$A \subset B . \tag{1.6}$$

In **Example 1.3**, the event "the value of the share is between "50\$ and 75\$" is given by the set $\left[50,75\right]$.

2 PROBABILITY SPACE

Given a sample space Ω, the set of all possible events will be noted by $\Im$, supposed to have the structure of a σ*-field* or a σ*-algebra.*

Definition 2.1 *The family* $\Im$ *of subsets of* Ω *is called a* σ*-field or a* σ*-algebra iff the following conditions are satisfied*:

(i) $\Omega,\varnothing$ *belong* to $\Im$,

(ii) Ω *is stable under denumerable intersection*:

$$A_n \in \Im, \forall n \geq 1 \Rightarrow \bigcap_{n\geq 1} A_n \in \Im, \tag{2.1}$$

(iii) $\Im$ *is stable for the complement set operation*

$$A \in \Im \Rightarrow A^c \in \Im, \ A^c = \Omega - A. \tag{2.2}$$

Then, using the well-known de Morgan's laws saying that

$$\left(\bigcup_{n\geq 1} A_n\right)^c = \bigcap_{n\geq 1} A_n^c, \ \left(\bigcap_{n\geq 1} A_n\right)^c = \bigcup_{n\geq 1} A_n^c, \tag{2.3}$$

it is easy to prove that a σ*-algebra* $\Im$ is also stable under denumerable union:

$$A_n \in \Im, \forall n \geq 1 \Rightarrow \bigcup_{n\geq 1} A_n \in \Im. \tag{2.4}$$

Any couple $(\Omega,\Im)$ where $\Im$ is a σ-algebra is called a *measurable space*.

The next definition concerning the concept of *probability measure* or simply *probability* is an idealization of the concept of the *frequency* of an event. Let us consider a random experiment called E with which is associated the couple

$(\Omega,\Im)$; if the set A belongs to $\Im$ and if we can repeat the experiment E n times, under the same conditions of environment, we can count how many times A occurs. If $n(A)$ represents this number of occurrences, the *frequency* of the event A is defined as

$$f(A)=\frac{n(A)}{n}. \tag{2.5}$$

In general, this number tends to become stable for large values of n.
The notion of frequency satisfies the following elementary properties:

(i) $(A,B\in\Im, A\cap B=\varnothing \Rightarrow f(A\cup B)=f(A)+f(B),$ (2.6)

(ii) $f(\Omega)=1$, (2.7)

(iii) $A,B\in\Im,\Rightarrow f(A\cup B)=f(A)+f(B)-f(A\cap B),$ (2.8)

(iv) $A\in\Im\Rightarrow f(A^c)=1-f(A).$ (2.9)

To have a useful mathematical model for the theoretical idealization of the notion of frequency, we now introduce the following definition.

Definition 2.2 a) *The triplet* $(\Omega,\Im,P)$ *is called a probability space if* Ω *is a non-void set of elements,* $\Im$ *a* σ*-algebra of subsets of* Ω *and P an application from* $\Im$ *to* $[0,1]$ *such that:*

(i)
$$\begin{aligned}&(A_n,n\geq 1), A_n\in\Im, n\geq 1:\left(i\neq j\Rightarrow A_i\cap A_j=\phi\right)\\&\Rightarrow P\left(\bigcup_{n\geq 1}A_n\right)=\sum_{n=1}^{\infty}P(A_n)\ (\sigma-additivity\ of\ P),\end{aligned} \tag{2.10}$$

(ii) $P(\Omega)=1.$ (2.11)

b) *The application P satisfying conditions* (2.10) *and* (2.11) *is called a probability measure or simply probability.*

Remark 2.1 1) The sequence of events $(A_n,n\geq 1)$ satisfying the condition that

$$(A_n,n\geq 1), A_n\in\Im, n\geq 1: i\neq j\Rightarrow A_i\cap A_j=\varnothing \tag{2.12}$$

is called *mutually exclusive*.
2) The relation (2.11) assigns the value 1 for the probability of the entire sample space Ω. There may exist events A' strictly subsets of Ω such that

$$P\left(A'\right)=1. \tag{2.13}$$

In this case, we say that A is *almost sure* or that the statement defining A is true *almost surely* (in short a.s.) or holds for almost all ω.
From axioms (2.10) and (2.11), we can deduce the following properties:

Property 2.1 (i) If $A,B\in\Im,$ then

$$P(A\cup B)=P(A)+P(B)-P(A\cap B). \tag{2.14}$$

(ii) If $A\in\Im,$ then

$$P(A^c) = 1 - P(A). \tag{2.15}$$

(iii) $P(\varnothing) = 0.$ (2.16)

(iv) If $(B_n, n \geq 1)$ is a sequence of disjoint elements of $\Im$ forming a partition of Ω, then for all A belonging to $\Im$,

$$P(A) = \sum_{n=1}^{\infty} P(A \cap B_n). \tag{2.17}$$

(v) *Continuity property of P*: if $(A_n, n \geq 1)$ is an increasing (decreasing) sequence of elements of $\Im$, then

$$P\left(\bigcup_{n\geq 1} A_n\right) = \lim_n P(A_n); \left(P\left(\bigcap_{n\geq 1} A_n\right) = \lim_n P(A_n)\right). \tag{2.18}$$

Remark 2.2 a) Boole's inequality asserts that if $(A_n, n \geq 1)$ is a sequence of events, then

$$P\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n\geq 1} P(A_n). \tag{2.19}$$

b) From (2.14), it is clear that we also have

$$A \subset B \Rightarrow P(A) \leq P(B). \tag{2.20}$$

Example 2.1 a) *The discrete case*

When the sample space Ω is *finite* or *denumerable*, we can set

$$\Omega = \{\omega_1, ..., \omega_j, ...\} \tag{2.21}$$

and select for $\Im$ the set of all the subsets of Ω, represented by 2^Ω.

Any probability measure P can be defined with the following sequence:

$$(p_j, j \geq 1),\ p_j \geq 0, j \geq 1,\ \sum_{j\geq 1} p_j = 1 \tag{2.22}$$

so that

$$P(\{w_j\}) = p_j, j \geq 1. \tag{2.23}$$

On the probability space $(\Omega, 2^\Omega, P)$, the probability assigned for an arbitrary event A = $\{\omega_{k_1}, ..., \omega_{k_l}\}, k_j \geq 1, j = 1, ..., l, k_i \neq k_j$ if $i \neq j$ is given by

$$P(A) = \sum_{j=1}^{l} p_{k_j}. \tag{2.24}$$

b) *The continuous case*

Let Ω be the real set $\mathbb{R}$; It can be proven (Halmos (1974)) that there exists a minimal σ-algebra generated by the set of intervals:

$$\beta = \{(a,b),[a,b],[a,b),(a,b],a,b \in \mathbb{R}, a \le b\}. \quad (2.25)$$

It is called the *Borel* σ*-algebra* represented by β and the elements of β are called *Borel sets*.

Given a probability measure *P* on (Ω, β), we can define the real function *F*, called the distribution function related to *P*, as follows.

Definition 2.3 *The function F from* $\mathbb{R}$ *to* $[0,1]$ *defined by*

$$P((-\infty, x]) = F(x), x \in \mathbb{R} \quad (2.26)$$

is called the distribution function related to the probability measure P.

From this definition and the basic properties of *P*, we easily deduce that:

$$\begin{aligned} &P((a,b]) = F(b) - F(a),\ P((a,b)) = F(b-) - F(a), \\ &P([a,b)) = F(b-) - F(a-),\ P([a,b]) = F(b) - F(a-). \end{aligned} \quad (2.27)$$

Moreover, from (2.26), any function F from $\mathbb{R}$ to $[0,1]$ is a distribution function (in short d.f.) iff it is a non-decreasing function satisfying the following conditions:

F is right continuous at every point x_0,

$$\lim_{x \uparrow x_0} F(x) = F(x_0), \quad (2.28)$$

and moreover

$$\lim_{x \to +\infty} F(x) = 1, \lim_{x \to -\infty} F(x) = 0. \quad (2.29)$$

If the function *F* is derivable on $\mathbb{R}$ with *f* as derivative, we have

$$F(x) = \int_{-\infty}^{x} f(y)dy, x \in \mathbb{R}. \quad (2.30)$$

The function *f* is called the density function associated with the d.f. *F* and in the case of the existence of such a Lebesgue integrable function on $\mathbb{R}$, *F* is called *absolutely continuous*.

From the definition of the concept of integral, we can give the intuitive interpretation of *f* as follows; given the small positive real number Δx, we have

$$P(\{x, x + \Delta x\}) \approx f(x)\Delta x. \quad (2.31)$$

Using the Lebesgue-Stieltjes integral, it can be seen that it is possible to define a probability measure *P* on $(\mathbb{R}, \beta)$ starting from a d.f. *F* on $\mathbb{R}$ by the following definition of *P*:

$$P(A) = \int_A dF(x), \forall A \in \Im. \quad (2.32)$$

In the absolutely continuous case, we get

$$P(A) = \int_A f(y)dy. \quad (2.33)$$

Remark 2.3 In fact, it is also possible to define the concept of d.f. in the discrete case if we set, without loss of generality, on $(N_0, 2^{N_0})$, the measure P defined from the sequence (2.22). Indeed, if for every positive integer k, we set

$$F(k) = \sum_{j=1}^{k} p_j \tag{2.34}$$

and generally, for any real x,

$$F(x) = \begin{cases} 0, x \le 0, \\ F(k), x \in [k, k+1), \end{cases} \tag{2.35}$$

then, for any positive integer k, we can write

$$P(\{1,...,k\}) = F(k) \tag{2.36}$$

and so calculate the probability of any event.

3 RANDOM VARIABLES

Let us suppose the probability space $(\Omega, \Im, P)$ and the measurable space (E, ψ) are given.

Definition 3.1 *A random variable* (*in short r.v.*) *with values in E is an application X from* Ω *to E such that*

$$\forall B \in \psi : X^{-1}(B) \in \Im, \tag{3.1}$$

where $X^{-1}(B)$ *is called the inverse image of the set B defined by*

$$X^{-1}(B) = \{\omega : X(\omega) \in B\}, X^{-1}(B) \in \Im. \tag{3.2}$$

Particular cases

a) If $(E, \psi) = (\mathbb{R}, \beta)$, X is called a *real random variable*.

b) If $(E, \psi) = (\overline{\mathbb{R}}, \overline{\beta})$, where $\overline{\mathbb{R}}$ is the *extended real line* defined by $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ and $\overline{\beta}$ the *extended Borel* σ*-field* of $\overline{\mathbb{R}}$, that is the minimal σ-field containing all the elements of β and the extended intervals

$$\begin{aligned} &[-\infty, a), (-\infty, a], [-\infty, a], (-\infty, a), \\ &[a, +\infty), (a, +\infty], [a, +\infty], (a, +\infty), \ a \in \mathbb{R}, \end{aligned} \tag{3.3}$$

then X is called a *real extended value random variable*.

c) If $E = \mathbb{R}^n (n > 1)$ with the product σ-field $\beta^{(n)}$ of β, X is called an *n-dimensional real random variable*.

d) If $E = \overline{\mathbb{R}}^{(n)}$ (n>1) with the product σ-field $\overline{\beta}^{(n)}$ of $\overline{\beta}$, X is called a real *extended n-dimensional real random variable*.

A random variable X is called *discrete* or *continuous* according as X takes at most a denumerable or a non-denumerable infinite set of values.

Remark 3.1 In *measure theory*, the only difference is that condition (2.11) is no longer required and in this case the definition of a r.v. given above gives the notion of *measurable function*. In particular a measurable function from $(\mathbb{R},\beta)$ to $(\mathbb{R},\beta)$ is called a *Borel function*.

Let X be a real r.v. and let us consider, for any real x, the following subset of Ω: $\{\omega : X(\omega) \leq x\}$.
As, from relation (3.2),

$$\{\omega : X(\omega) \leq x\} = X^{-1}((-\infty, x]), \tag{3.4}$$

it is clear from relation (3.1) that this set belongs to the σ-algebra $\Im$. Conversely, it can be proved that the condition

$$\{\omega : X(\omega) \leq x\} \in \Im, \tag{3.5}$$

valid for every x belonging to a dense subset of $\mathbb{R}$, is sufficient for X being a real random variable defined on Ω. The probability measure P on $(\Omega, \Im)$ induces a probability measure μ on $(\mathbb{R},\beta)$ defined as

$$\forall B \in \beta : \mu(B) = P(\{\omega : X(\omega) \in B\}). \tag{3.6}$$

We say that μ is the induced probability measure on $(\mathbb{R},\beta)$, called the *probability distribution* of the r.v. X. Introducing the distribution function related to μ, we get the next definition.

Definition 3.2 *The distribution function of the r.v. X, represented by F_X, is the function from $\mathbb{R} \to [0,1]$ defined by*

$$F_X(x) = \mu((-\infty, x]) = P(\{\omega : X(\omega) \leq x\}). \tag{3.7}$$

In short, we write

$$F_X(x) = P(X \leq x). \tag{3.8}$$

This last definition can be extended to the multi-dimensional case with a r.v. X being an n-dimensional real vector: $X = (X_1, ..., X_n)$, a measurable application from $(\Omega, \Im, P)$ to $(\mathbb{R}^n, \beta^n)$.

Definition 3.3 *The distribution function of the r.v. $X = (X_1, ..., X_n)$, represented by F_X, is the function from $\mathbb{R}^n$ to $[0,1]$ defined by*

$$F_X(x_1,...,x_n) = P\left(\{\omega : X_1(\omega) \le x_1,...,X_n(\omega) \le x_n\}\right). \quad (3.9)$$

In short, we write

$$F_X(x_1,...,x_n) = P(X_1 \le x_1,...,X_n \le x_n). \quad (3.10)$$

Each component X_i (i=1,…,n) is itself a one-dimensional real r.v. whose d.f., called the *marginal d.f.*, is given by

$$F_{X_i}(x_i) = F_X(+\infty,...,+\infty,x_i,+\infty,...,+\infty). \quad (3.11)$$

The concept of random variable is *stable* under a lot of mathematical operations; so any Borel function of a r.v. X is also a r.v.
Moreover, if X and Y are two r.v., so are

$$\inf\{X,Y\},\sup\{X,Y\},X+Y,X-Y,X\cdot Y,\frac{X}{Y}, \quad (3.12)$$

provided, in the last case, that Y does not vanish.
Concerning the convergence properties, we must mention the property that, if $(X_n, n\ge 1)$ is a *convergent* sequence of r.v. $-$ that is, for all $\omega\in\Omega$, the sequence $(X_n(\omega))$ converges to $X(\omega)$ $-$, then the limit X is also a r.v. on Ω. This convergence, which may be called the *sure convergence*, can be weakened to give the concept of *almost sure* (in short a.s.) *convergence* of the given sequence.

Definition 3.4 *The sequence* $(X_n(\omega))$ *converges a.s. to* $X(\omega)$ *if*

$$P\left(\{\omega : \lim X_n(\omega) = X(\omega)\}\right) = 1. \quad (3.13)$$

This last notion means that the possible set where the given sequence does not converge is a *null set*, that is a set N belonging to $\Im$ such that

$$P(N) = 0. \quad (3.14)$$

In general, let us remark that, given a null set, it is not true that every subset of it belongs to $\Im$ but of course if it belongs to $\Im$, it is clearly a null set (see relation (2.20)).
To avoid unnecessary complications, we will suppose from now on that any considered probability space is *complete*, This means that all the subsets of a null set also belong to $\Im$ and thus that their probability is zero.

4 INTEGRABILITY, EXPECTATION AND INDEPENDENCE

Let us consider a complete measurable space $(\Omega,\Im,\mu)$ and a real measurable variable X defined on Ω. To any set A belonging to $\Im$, we associate the r.v. I_A, called the indicator of A, defined as

$$I_A(\omega)=\begin{cases}1, \omega\in A,\\ 0, \omega\notin A.\end{cases} \tag{4.1}$$

If there exists partition $(A_n, n\geq 1)$ with all its sets measurable such that

$$\omega\in A_n \Rightarrow X(\omega)=a_n (a_n\in\mathbb{R}), n\geq 1, \tag{4.2}$$

then X is called a *discrete* variable. If moreover, the partition is finite, it is said to be *finite*. It follows that we can write X in the following form:

$$X(\omega)=\sum_{n=1}^{\infty} a_n I_{A_n}(\omega). \tag{4.3}$$

Definition 4.1 *The integral of the discrete variable X is defined by*

$$\int_\Omega X d\mu=\sum_{n=1}^{\infty} a_n \mu(A_n), \tag{4.4}$$

provided that this series is absolutely convergent.

Of course, if X is integrable, we have the integrability of $|X|$ too and

$$\int_\Omega |X| d\mu=\sum_{n=1}^{\infty} |a_n| \mu(A_n). \tag{4.5}$$

To define in general the integral of a measurable function X, we first restrict ourselves to the case of a non-negative measurable variable X for which we can construct a monotone sequence $(X_n, n\geq 1)$ of discrete variables converging to X as follows:

$$X_n(\omega)=\sum_{k=1}^{\infty}\frac{k}{2^n} I_{\left\{\omega: \frac{k}{2^n}\leq X<\frac{k+1}{2^n}\right\}}. \tag{4.6}$$

Since for each n,

$$\begin{aligned}&X_n(\omega)\leq X_{n+1}(\omega),\\ &0\leq X(\omega)-X_n(\omega)\leq\frac{1}{2^n},\end{aligned} \tag{4.7}$$

the sequence $(X_n, n\geq 1)$ of discrete variables converges monotonically to X on Ω.

Definition 4.2 *The non-negative measurable variable X is integrable on Ω iff the elements of the sequence $(X_n, n\geq 1)$ of discrete variables defined by relation* (4.6) *are integrable and if the sequence $\left(\int_\Omega X_n dP\right)$ converges.*

From this last definition, it follows that

$$E(X)=\lim E(X_n), \tag{4.8}$$

where

$$\int_{\Omega} X_n(\omega) d\mu = \sum_{k=1}^{\infty} \frac{k}{2^n} \mu\left(I_{\left\{\omega: \frac{k}{2^n} \le X < \frac{k+1}{2^n}\right\}} \right). \tag{4.9}$$

To extend the last definition without the non-negativity condition on *X*, let us introduce for an arbitrary variable *X*, the variables X^+ and X^- defined by

$$X^+(\omega) = \sup\{X(\omega), 0\},\ X^-(\omega) = -\inf\{X(\omega), 0\}, \tag{4.10}$$

so that

$$X = X^+ - X^-. \tag{4.11}$$

Definition 4.3 *The measurable variable X is integrable on* Ω *iff the non-negative variables* X^+ *and* X^- *defined by relation* (4.10) *are integrable and in this case*

$$\int_{\Omega} X d\mu = \int_{\Omega} X^+ d\mu - \int_{\Omega} X^- d\mu. \tag{4.12}$$

Remark 4.1 a) If the integral of *X* does not exist, it may however happen that

$$\int_{\Omega} X^+ d\mu < \infty \left(\int_{\Omega} X^- d\mu < \infty \right), \int_{\Omega} X^- d\mu = \infty \left(\int_{\Omega} X^+ d\mu = \infty \right). \tag{4.13}$$

In these two cases, we say that the integral of *X* is infinite; more precisely, we have

$$\int_{\Omega} X d\mu = -\infty \left(\int_{\Omega} X d\mu = +\infty \right). \tag{4.14}$$

If *A* is an element of the σ-algebra $\Im$, the integral on *A* is simply defined by

$$\int_{A} X d\mu = \int_{A} X I_A d\mu. \tag{4.15}$$

Of course, *X* being a non-negative measurable variable with an infinite integral, it means that the approximation sequence (4.6) diverges to $+\infty$ for almost all ω.

Now let us consider a probability space $(\Omega, \Im, P)$ and a real random variable *X* defined on Ω. In this case, the concept of integrability is designed by *expectation* represented by

$$E(X) = \int_{\Omega} X dP \left(= \int X dP \right), \tag{4.16}$$

provided that this integral exists. The computation of the integral

$$\int_{\Omega} X dP \left(= \int X dP \right) \tag{4.17}$$

can be done using the induced measure μ on $(\mathbb{R}, \beta)$,defined by relation (3.6) and then using the distribution function *F* of *X*. Indeed, we can write

$$E(X)\left(=\int_{\Omega} XdP\right)=\int_{R} Xd\mu, \tag{4.18}$$

and if F_X is the d.f. of X, it can be shown that

$$E(X)=\int_{R} xdF_X(x), \tag{4.19}$$

this last integral being a Lebesgue-Stieltjes integral. Moreover, if F_X is absolutely continuous with f_X as density, we get

$$E(X)=\int_{-\infty}^{+\infty} xf_x(x)dx. \tag{4.20}$$

If g is a Borel function, we also have (see for example Chung (2000), Royden (1963), Loeve (1963))

$$E(g(X))=\int_{-\infty}^{+\infty} g(x)dF_X \tag{4.21}$$

and with a density for X,

$$E(g(X))=\int_{-\infty}^{+\infty} g(x)f_X(x)dx. \tag{4.22}$$

The most important properties of the expectation are given in the next proposition.

Proposition 4.1 (i) *Linearity property of the expectation: If X and Y are two integrable r.v. and a,b two real numbers, then the r.v. aX+bY is also integrable and*

$$E(aX+bY)=aE(X)+bE(Y). \tag{4.23}$$

(ii) *If* $(A_n, n\geq 1)$ *is a partition of* Ω, *then*

$$E(X)=\sum_{n=1}^{\infty}\int_{A_n} XdP. \tag{4.24}$$

(iii) *The expectation of a non-negative r.v. is non-negative.*
(iv) *If X and Y are integrable r.v., then*

$$X\leq Y \Rightarrow E(X)\leq E(Y). \tag{4.25}$$

(v) *If X is integrable, so is* $|X|$ *and*

$$|E(X)|\leq E|X|. \tag{4.26}$$

(vi) *Dominated convergence theorem (Lebesgue: Let* $(X_n, n\geq 1)$ *be a sequence of r.v. converging a.s. to the r.v. X integrable, then all the r.v.* X_n *are integrable and moreover*

$$\lim E(X_n)=E(\lim X_n)\ (=E(X)). \tag{4.27}$$

(vii) *Monotone convergence theorem (Lebesgue): Let* $(X_n, n \geq 1)$ *be a non-decreasing sequence of non-negative r.v; then relation* (4.27) *is still true provided that* $+\infty$ *is a possible value for each member.*
(viii) *If the sequence of integrable r.v.* $(X_n, n \geq 1)$ *is such that*

$$\sum_{n=1}^{\infty} E\left(|X_n|\right) < \infty, \tag{4.28}$$

then the random series $\sum_{n=1}^{\infty} X_n$ *converges absolutely a.s. and moreover*

$$\sum_{n=1}^{\infty} E(X_n) = E\left(\sum_{n=1}^{\infty} X_n\right) \ (= E(X)), \tag{4.29}$$

where the r.v. is defined as the sum of the convergent series.

Given a r.v. *X*, *moments* are special cases of expectation.

Definition 4.4 *Let a be a real number and r a positive real number, then the expectation*

$$E\left(|X-a|^r\right) \tag{4.30}$$

is called the absolute moment of X, of order r, centred on a.

The moments are said to be centred moments of order *r* if $a=E(X)$. In particular, for $r=2$, we get the *variance* of *X* represented by σ^2 $(\text{var}(X))$,

$$\sigma^2 = E\left(|X-m|^2\right). \tag{4.31}$$

Remark 4.2 From the linearity of the expectation (see relation (4.23)), it is easy to prove that

$$\sigma^2 = E(X^2) - (E(X))^2, \tag{4.32}$$

and so

$$\sigma^2 \leq E(X^2), \tag{4.33}$$

and more generally, it can be proven that the variance is the smallest moment of order 2 whatever the number *a* is.

The next property recalls inequalities for moments.

Proposition 4.2 (*Inequalities of Hölder and Minkowski*) (i) *Let X and Y be two r.v. such that* $|X|^p, |Y|^q$ *are integrable with*

$$1 < p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1, \tag{4.34}$$

then:

$$\left|E(XY)\right| \le \left(E\left(\left|X^p\right|\right)\right)^{1/p} \cdot \left(E\left(\left|Y^q\right|\right)\right)^{1/q}. \qquad (4.35)$$

(ii) *Let X and Y be two r.v. such that* $\left|X\right|^p, \left|Y\right|^p, 1 \le p < \infty$, *are integrable, then*

$$E\left(\left|X+Y\right|^p\right)^{1/p} \le \left(E\left(\left|X\right|^p\right)\right)^{1/p} + \left(E\left(\left|Y\right|^p\right)\right)^{1/p}. \qquad (4.36)$$

If *p=2* in the first part of this last proposition, then relation (4.36) gives the *Cauchy-Schwarz inequality*

$$\left|E(XY)\right| \le \left(E\left(\left|X\right|^2\right)\right)^{1/2} \cdot \left(E\left(\left|Y\right|^2\right)\right)^{1/2}. \qquad (4.37)$$

The last fundamental concept we will now introduce in this section is that of *stochastic independence*, or more simply *independence*.

Definition 4.5 *The events* $A_1, ..., A_n, (n > 1)$ *are stochastically independent or independent iff*

$$\forall m = 2,...,n, \forall n_k = 1,...,n : n_1 \ne n_2 \ne \cdots \ne n_k : P\left(\bigcap_{k=1}^{m} A_{n_k}\right) = \prod_{k=1}^{m} P(A_{n_k}). \qquad (4.38)$$

For *n=2*, relation (4.38) reduces to

$$P(A_1 \cap A_2) = P(A_1)P(A_2). \qquad (4.39)$$

Let us remark that piecewise independence of the events $A_1, ..., A_n, (n > 1)$ does not necessarily imply the independence of these sets and thus not the stochastic independence of these *n* events. As a counter example, let us suppose we drew a ball from an urn containing four balls called b_1, b_2, b_3, b_4 and let us consider the three following events:

$$A_1 = \{b_1, b_2\}, A_2 = \{b_1, b_3\}, A_3 = \{b_1, b_4\}. \qquad (4.40)$$

Then assuming that the probability of having one ball is $1/4$, we get

$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = \frac{1}{4}, \qquad (4.41)$$

but as

$$P(A_1 \cap A_2 \cap A_3) = \frac{1}{4} \qquad (4.42)$$

too, we do not have the relation

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3), \qquad (4.43)$$

and so we have proved that independence in pairs does not imply the independence of these three events.
We will now extend the concept of independence to random variables.

Definition 4.6 (i) *The n real r.v. $X_1, X_2, ..., X_n$ defined on the probability space $(\Omega, \Im, P)$ are said to be stochastically independent, or simply independent, iff for any Borel sets $B_1, B_2, ..., B_n$, we have*

$$P\left(\bigcap_{k=1}^{n}\{\omega : X_k(\omega) \in B_k\}\right) = \prod_{k=1}^{n} P\left(\{\omega : X_k(\omega) \in B_k\}\right). \tag{4.44}$$

(ii) *For an infinite family of r.v., independence means that the members of every finite subfamily are independent. It is clear that if $X_1, X_2, ..., X_n$ are independent, so are the r.v. $X_{i_1}, ..., X_{i_k}$ with $i_1 \neq \cdots \neq i_k$, $i_k = 1, ..., n, k = 2, ..., n$.*

From relation (4.44), we find that

$$P(X_1 \le x_1, ..., X_n \le x_n) = P(X_1 \le x_1) \cdots P(X_n \le x_n), \forall (x_1, ..., x_n) \in \mathbb{R}^n. \tag{4.45}$$

If the functions $F_X, F_{X_1}, ..., F_{X_n}$ are the distribution functions of the r.v. $X = (X_1, ..., X_n), X_1, ..., X_n$, we can write the preceding relation under the form

$$F_X(x_1, ..., x_n) = F_{X_1}(x_1) \cdot \cdots \cdot F_{X_n}(x_n), \forall (x_1, ..., x_n) \in \mathbb{R}^n. \tag{4.46}$$

It can be shown that this last condition is also sufficient for the independence of $X = (X_1, ..., X_n), X_1, ..., X_n$. If these d.f. have densities $f_X, f_{X_1}, ..., f_{X_n}$, relation (4.46) is equivalent to

$$f_X(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n), \forall (x_1, ..., x_n) \in \mathbb{R}^n. \tag{4.47}$$

In case of the integrability of the *n* real r.v $X_1, X_2, ..., X_n$, a direct consequence of relation (4.46) is that we have a very important property for the expectation of the product of *n* independent r.v.:

$$E\left(\prod_{k=1}^{n} X_k\right) = \prod_{k=1}^{n} E(X_k). \tag{4.48}$$

The notion of independence gives the possibility to prove the result called the *strong law of large numbers* which says that if $(X_n, n \ge 1)$ is a sequence of integrable independent and identically distributed r.v., then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} E(X). \tag{4.49}$$

The next section will present the most useful distribution functions for stochastic modelling.

5 MAIN DISTRIBUTION PROBABILITIES

Here we shall restrict ourselves to presenting the principal distribution probabilities related to real random variables.

5.1 The Binomial Distribution

Let us consider a random experiment E such that only two results are possible: a "success"(S) with probability p and a "failure (F) with *probability* $q = 1 - p$. If n independent trials are made in exactly the same experimental environment, the total number of trials in which the event S occurs may be represented by a random variable X whose distribution $(p_i, i = 0,...,n)$ with

$$p_i = P(X = i),\ i = 1,...,n \tag{5.1}$$

is called a *binomial distribution* with parameters (n,p). From basic axioms of probability theory seen before, it is easy to prove that

$$p_i = \binom{n}{i} p^i q^{n-i},\ i = 0,...,n, \tag{5.2}$$

a result from which we get

$$E(X) = np, \operatorname{var}(X) = npq. \tag{5.3}$$

The *characteristic function* and the *generating function, when it exists,* of X respectively defined by

$$\begin{aligned} \varphi_X(t) &= E(e^{itX}), \\ g_X(t) &= E(e^{tX}) \end{aligned} \tag{5.4}$$

are given by

$$\begin{aligned} \varphi_X(t) &= (pe^{it} + q)^n, \\ g_X(t) &= (pe^t + q)^n. \end{aligned} \tag{5.5}$$

Example 5.1 (*The Cox and Rubinstein financial model*) Let us consider a financial asset observed on n successive discrete time periods so that at the beginning of the first period, from time 0 to time 1, the asset starts from value S_0 and has at the end of this period only two possible values, uS_0 and dS_0 ($0<d<1, u>1$) respectively with probabilities p and q=1-p. The asset has the same type of evolution on each period and independently of the past. In period i, from time i–1 to time i, let us associate the r.v. $\xi_i, i = 1,...,n$ defined as follows:

$$\xi_i = \begin{cases} 1, \text{ with probability } p, \\ 0, \text{ with probability } q. \end{cases} \tag{5.6}$$

The value of the asset at the end of period n is given by the r.v. Y_n defined as

$$Y_n = u^{X_n} d^{n-X_n} \tag{5.7}$$

with

$$X_n = \xi_1 + \cdots + \xi_n. \tag{5.8}$$

It is clear that the r.v. X_n has a binomial distribution of parameters (n,p) and consequently, we get the distribution probability of Y_n:

$$P(Y_n = u^j d^{n-j} S_0) = \binom{n}{i} p^i q^{n-i},\ i = 0,...,n. \tag{5.9}$$

This distribution is currently used in the financial model of Cox, Ross and Rubinstein (1979) developed in Chapter 5.

5.2 The Poisson Distribution

If X is a r.v. with values in $\mathbb{N}$ so that the probability distribution is given by

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}, i = 0,1,... \tag{5.10}$$

where λ is a strictly positive constant, X is called a *Poisson variable of parameter* λ. This is one of the most important distributions for all applications. For example if we consider an insurance company looking at the total number of claims in one year, this variable often may be considered as a Poisson variable. Basic parameters of this Poisson distribution are given here:

$$\begin{aligned} &E(X) = \lambda,\ \text{var}(X) = \lambda, \\ &\varphi_X(t) = e^{\lambda(e^{it}-1)},\ g_X(t) = e^{\lambda(e^t-1)}. \end{aligned} \tag{5.11}$$

A remarkable result is that the Poisson distribution is the limit of a binomial distribution of parameters (n,p) if n tends to $+\infty$ and p to 0 so that np converges to λ.

The Poisson distribution is often used for the occurrence of rare events. For example if an insurance company wants to hedge the hurricane risk in the States and if we know that the mean number of hurricanes per year is 3, the adjustment of the r.v. X defined as the number of hurricanes per year with a Poisson distribution of parameter $\lambda = 3$ gives the following results:

$P(X=0)=0.0498$, $P(X=1)=0.1494$, $P(X=2)=0.2240$, $P(X=3)=0.2240$,
$P(X=4)=0.1680$, $P(X=5)=0.1008$, $P(X=6)=0.0504$, $P(X>6)=0.0336$.

So the probability that the company has to hedge two or three hurricanes per year is 0.4480.

5.3 The Normal (or Laplace-Gauss) Distribution

The real r.v. X has a normal (or Laplace-Gauss) distribution of parameters $(\mu,\sigma^2), \mu \in \mathbb{R}, \sigma^2 > 0$, if its density function is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}. \tag{5.12}$$

From now on, we will use the notation $X \prec N(\mu,\sigma^2)$. The main parameters of this distribution are

$$E(X) = \mu, \ \operatorname{var}(X) = \sigma^2,$$
$$\varphi_X(t) = \exp\left(i\mu t - \frac{\sigma^2 t^2}{2}\right), \ g_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right). \tag{5.13}$$

If $\mu = 0, \ \sigma^2 = 1$, the distribution of X is called a *reduced* or *standard normal distribution*. In fact, if X has a normal distribution $(\mu, \sigma^2), \mu \in R, \sigma^2 > 0$, then the so-called reduced r.v. Y defined by

$$Y = \frac{X - \mu}{\sigma} \tag{5.14}$$

has a standard normal distribution, thus from (5.13) with mean 0 and variance 1. Let Φ be the distribution function of the standard normal distribution; it is possible to express the distribution function of any normal r.v. X of parameters $(\mu, \sigma^2), \mu \in \mathbb{R}, \sigma^2 > 0$ as follows:

$$F_X(x) = P(X \le x) = P\left(\frac{X - \mu}{\sigma} \le \frac{x - \mu}{\sigma}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right). \tag{5.15}$$

Also from the numerical point of view, it suffices to know numerical values for the standard distribution. From relation (5.15), we also deduce that

$$f_X(x) = \frac{1}{\sigma}\Phi'\left(\frac{x - \mu}{\sigma}\right), \tag{5.16}$$

where of course from (5.12)

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{5.17}$$

From the definition of Φ, we have

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, \ x \in \mathbb{R} \tag{5.18}$$

and so

$$\Phi(-x) = 1 - \Phi(x), x > 0, \tag{5.19}$$

and consequently, for X normally distributed with parameters (0,1), we get

$$P(|X| \le x) = \Phi(x) - \Phi(-x) = 2\Phi(x) - 1, \ x > 0. \tag{5.20}$$

In particular, let us mention the following numerical results:

$$\begin{aligned}
&P\left(|X - m| \le \frac{2}{3}\sigma\right) = 0.4972(\approx 50\%),\\
&P(|X - m| \le \sigma) = 0.6826(\approx 68\%),\\
&P(|X - m| \le 2\sigma) = 0.9544(\approx 95\%),\\
&P(|X - m| \le 3\sigma) = 0.9974(\approx 99\%).
\end{aligned} \tag{5.21}$$

Remark 5.1: *Numerical computation of the d.f.* Φ For applications in finance, for example the Black Scholes (1973) model for option pricing (see Chapter 5), we will need the following numerical approximation method for computing Φ *with seven exact decimals* instead of the four given by the standard statistical tables:

$$\begin{aligned}
&1)\ x>0\ :\\
&\Phi(x)\approx 1-\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}(b_1c+\cdots+b_5c^5),\\
&c=\frac{1}{1+px},\\
&p=0{,}2316419, b_1=0{,}319381530,\\
&b_2=-0{,}356563782, b_3=1{,}781477937,\\
&b_4=-1{,}821255978, b_5=1{,}330274429,\\
&2) x<0\ :\\
&\Phi(x)=1-\Phi(-x).
\end{aligned} \tag{5.22}$$

The normal distribution is one of the most often used distributions, by virtue of the *Central Limit Theorem* which says that if $(X_n, n\geq 1)$ is a sequence of independent identically distributed (in short i.i.d.) r.v. with mean m and variance σ^2, then the sequence of r.v. defined by

$$\frac{S_n-nm}{\sigma\sqrt{n}} \tag{5.23}$$

with

$$S_n=X_1+\cdots+X_n,\ \ n>0 \tag{5.24}$$

converges in law to a standard normal distribution. This means that the sequence of the distribution functions of the variables defined by (5.21) converges to Φ. This theorem was used by the Nobel Prize winner H. Markowitz (1959) to justify that the return of a diversified portfolio of assets has a normal distribution. As a particular case of the Central Limit Theorem, let us mention *de Moivre's theorem* starting with

$$X_n=\begin{cases}1, \text{ with prob. } p,\\ 0, \text{ with prob. } 1-p,\end{cases} \tag{5.25}$$

so that, for each *n,* the r.v. defined by relation (5.22) has a binomial distribution with parameters (*n,p*). By applying now the Central Limit Theorem, we get the following result:

$$\frac{S_n-np}{\sqrt{np(1-p)}}\xrightarrow[n\to+\infty]{law}N(0,1), \tag{5.26}$$

called de Moivre's result.

5.4 The Log-Normal Distribution

If the normal distribution is the most frequently used, it is nevertheless true that it could not be used for example to model the time evolution of a financial asset like a share or a bond, as the minimal value of these assets is 0 and so the support of their d.f. is the real half-line $[0,+\infty)$. One possible solution is to consider the *truncated normal distribution* to be defined by setting all the probability mass of the normal distribution on the negative real half-line on the positive side, but then all the interesting properties of the normal distribution are lost.
Also, in order to have a better approach to some financial market data, we have to introduce the *log-normal distribution*. The real non-negative random variable X has a *lognormal distribution* of parameters μ,σ – and we will write $X \prec LN(\mu,\sigma)$ – if the r.v. log*X* has a normal distribution with parameters μ,σ^2. Consequently, the density function of *X* is given by

$$f_X(x)=\begin{cases} 0, x\le 0, \\ \dfrac{1}{x\sqrt{2\pi}\sigma}e^{-\frac{(\log x-\mu)^2}{2\sigma^2}}, x\ge 0. \end{cases} \tag{5.27}$$

Indeed, we can write

$$P(X\le x)=P(\log X\le \log x), \tag{5.28}$$

and so

$$F_X(x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\log x} e^{\frac{(t-\mu)^2}{2\sigma^2}}dt\left(=\Phi\left(\frac{\log x-\mu}{\sigma}\right)\right), \tag{5.29}$$

and after the change of variable *t*=log*x,* we get the relation (5.27). Let us remark that the relation (5.29) is the most useful one for the computation of the d.f. of *X* with the help of the normal d.f. For the density function, we can also write

$$f_X(x)=\frac{1}{\sigma x}\Phi\left(\frac{\log x-\mu}{\sigma}\right). \tag{5.30}$$

The basic parameters of this distribution are given by

$$\begin{aligned} &E(X)=e^{\mu+\frac{\sigma^2}{2}}, \\ &\operatorname{var}(X)=e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right), \\ &E\left(X^r\right)=e^{r\left(\mu+\frac{\sigma^2}{2}\right)}. \end{aligned} \tag{5.31}$$

Let us say that the log-normal distribution has no generating function and that the characteristic function has no explicit form. When $\sigma < 0.3$, some authors recommend a normal approximation with parameters (μ, σ^2).

The normal distribution is *stable* under the addition of independent random variables; this property means that the sum of *n* independent normal r.v. is still normal. That is no longer the case with the log-normal distribution which is stable under *multiplication*, which means that for two independent log-normal r.v. X_1, X_2, we have

$$X_i \prec LN(\mu_i, \sigma_i), i = 1,2 \Rightarrow X_1 \times X_2 \prec LN\left(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right). \tag{5.32}$$

5.5 The Negative Exponential Distribution

The non-negative r.v. X has a *negative exponential distribution* (or simply *exponential distribution*) of parameter λ if its density function is given by

$$f_X(x) = \lambda e^{-\lambda x}, x \geq 0, \tag{5.33}$$

where λ is a strictly positive real number. By integration, we get the explicit form of the exponential distribution function

$$F_X(x) = 1 - e^{-\lambda x}, x \geq 0. \tag{5.34}$$

Of course, F_X is null for negative values of x. The basic parameters are

$$\begin{aligned} &E(X) = \frac{1}{\lambda}, \ \operatorname{var} X = \frac{1}{\lambda^2}, \\ &\varphi_X(t) = \frac{1}{1 - i\dfrac{t}{\lambda}}, \ g_X(t) = \frac{1}{1 - \dfrac{t}{\lambda}}, t < \lambda. \end{aligned} \tag{5.35}$$

In fact, this distribution is the first to be used in reliability theory.

5.6 The Multidimensional Normal Distribution

Let us consider an n-dimensional real r.v. X represented as a column vector of its *n* components $X = (X_1, ..., X_n)'$. Its d.f. is given by

$$F_X(x_1, ..., x_n) = P(X_1 \leq x_1, ..., X_n \leq x_n). \tag{5.36}$$

If the density function of X exists, the relations between the d.f. and the density function are

$$\begin{aligned} &f_X(x_1, ..., x_n) = \frac{\partial^n F_X}{\partial x_1 ... \partial x_n}(x_1, ..., x_n), \\ &F_X(x_1, ..., x_n) = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} f_X(\xi_1, ..., \xi_n) d\xi_1, ..., d\xi_n. \end{aligned} \tag{5.37}$$

For the principal parameters we will use the following notation:

$$\begin{aligned}&E(X_k)=\mu_k, k=1,...,n,\\&E((X_k-\mu_k)(X_l-\mu_l))=\sigma_{kl}, k,l=1,...,n,\\&E((X_k-\mu_k))^2=\sigma_k^2, k=1,...,n,\\&\rho_{kl}=\frac{E((X_k-\mu_k)(X_l-\mu_l))}{\sqrt{E((X_k-\mu_k)^2)E((X_k-\mu_k)^2)}}\left(=\frac{\sigma_{kl}}{\sigma_k\sigma_l}\right), k,l=1,...,n.\end{aligned} \tag{5.38}$$

The parameters σ_{kl} are called the *covariances* between the r.v. X_k and X_l and the parameters ρ_{kl}, the *correlation coefficients* between the r.v. X_k and X_l. It is well known that the correlation coefficient ρ_{kl} measures a certain linear dependence between the two r.v. X_k and X_l. More precisely, if it is equal to 0, there is no such dependence and the two variables are called *uncorrelated*; for the values +1 and –1 this dependence is certain.

With matrix notation, the $n\times n$ matrix

$$\mathbf{\Sigma}_X=\left[\sigma_{ij}\right] \tag{5.39}$$

is called the *variance –covariance* matrix of X. The *characteristic function* of X is defined as:

$$\varphi_X(t_1,...,t_n)=E\left(e^{i(t_1X_1+..+t_nX_n)_1}\right)\left(=E\left(e^{i\mathbf{t}'X}\right)\right). \tag{5.40}$$

Let $\boldsymbol{\mu},\mathbf{\Sigma}$ be respectively an n-dimensional real vector and an $n\times n$ positive definite matrix. The n-dimensional real r.v. X has a *non-degenerated n-dimensional normal distribution* with parameters $\boldsymbol{\mu},\mathbf{\Sigma}$ if its density function is given by

$$f_X(\mathbf{x})=\frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det\mathbf{\Sigma}}}e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}, \mathbf{x}\in\mathbb{R}^n. \tag{5.41}$$

Then, it can be shown by integration that parameters $\boldsymbol{\mu},\mathbf{\Sigma}$ are indeed respectively the *mean vector* and the *variance-covariance matrix* of X. As usual, we will use the notation: $X\prec N_n(\boldsymbol{\mu},\mathbf{\Sigma})$.

The characteristic function of X is given by

$$\varphi_X(\mathbf{t})=e^{i\boldsymbol{\mu}'\mathbf{t}-\frac{1}{2}\mathbf{t}'\mathbf{\Sigma}\mathbf{t}}. \tag{5.42}$$

The main fundamental properties of the n-dimensional normal distribution are:

-every subset of k r.v. of the set $\{X_1,\dots,X_n\}$ has also a k-dimensional distribution which is also normal;

-the multi-dimensional normal distribution is *stable* under linear transformations of X;

-the multi-dimensional normal distribution is *stable* for addition of random variables, which means that if $X_k \prec N_n(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k), k = 1,...,m$ and if these m random vectors are independent, then

$$X_1 + \cdots + X_m \prec N_n(\boldsymbol{\mu}_1 + \cdots + \boldsymbol{\mu}_m, \boldsymbol{\Sigma}_1 + \cdots + \boldsymbol{\Sigma}_m). \tag{5.43}$$

Particular case: *the two-dimensional normal distribution*
In this case, we have

$$\boldsymbol{\mu} = (\mu_1, \mu_2)', \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}, \rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2},$$

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\varsigma_1 \sigma_2} \\ -\frac{\rho}{\varsigma_1 \sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}, \det \boldsymbol{\Sigma} = \sigma_1 \sigma_2 \sqrt{1 - \rho^2}. \tag{5.44}$$

From the first main fundamental properties of the n-dimensional normal distribution given above, we have

$$X_k \prec N_1(\mu_k, \sigma_k^2), k = 1, 2. \tag{5.45}$$

For the special degenerated case of $|\rho| = 1$, it can be proved that

$$\begin{aligned} \rho = 1 &: \frac{X_2 - \mu_2}{\sigma_2} = \frac{X_1 - \mu_1}{\sigma_1}, \\ \rho = -1 &: \frac{X_2 - \mu_2}{\sigma_2} = -\frac{X_1 - \mu_1}{\sigma_1}, \end{aligned} \tag{5.46}$$

relations meaning that in this case, all the probability mass in the plan lies on a straight line so the two random variables X_1,X_2 are perfectly dependent with probability 1.
To finish this section, let us recall the well-known property saying that two independent r.v. are uncorrelated but the converse is not true except for the normal distribution.

6 CONDITIONING (FROM INDEPENDENCE TO DEPENDENCE)

6.1 Conditioning: Introductory Case

Let us begin to recall briefly the concept of *conditional probability*. Let $(\Omega, \Im, P)$ be a probability space and let A, B be elements of $\Im$ and look at the number of occurrences of the event A whenever B has already been observed in a sequence of n trials of our experiment. We shall call this number $n(A|B)$.
In terms of frequency of events defined by relation (2.5), we have

$$n\left(A\middle|B\right)=\frac{n(A\cap B)}{n(B)}, \tag{6.1}$$

provided that $n(B)$ is different from 0. Dividing by n the two members of relation (6.1), we get

$$\frac{n\left(A\middle|B\right)}{n}=\frac{\frac{n(A\cap B)}{n}}{\frac{n(B)}{n}}. \tag{6.2}$$

In terms of frequencies, we get

$$f\left(A\middle|B\right)=\frac{f(A\cap B)}{f(B)}. \tag{6.3}$$

From the experimental interpretation of the concept of probability of an event seen in section 2, we can now define the *conditional probability of A given B* as

$$P\left(A\middle|B\right)=\frac{P(A\cap B)}{P(B)}, P(B)>0. \tag{6.4}$$

If the events A and B are independent, from relation (4.39), we get

$$P\left(A\middle|B\right)=P(A), \tag{6.5}$$

a relation meaning that, in case of independence, the conditional probability of set A does not depend on the given set B. As the independence of sets A and B is equivalent to the independence of sets A and B^c, we also have:

$$P\left(A\middle|B^c\right)=P(A). \tag{6.6}$$

The notion of conditional probability is very useful for computing probabilities of a product of *dependent* events A and B not satisfying relation (4.39). Indeed from relations (6.4) and (6.6), we can write

$$P(A\cap B)=P(A)P\left(A\middle|B\right)=P(B)P\left(B\middle|A\right). \tag{6.7}$$

More generally, for n events $A_1,\dots,A_n$, we get the so-called "theorem of compound probability":

$$P\left(\bigcap_{k=1}^{n}A_k\right)=P(A_1)P\left(A_2\middle|A_1\right)\cdots P\left(A_n\middle|A_1\cap A_2\dots\cap A_{n-1}\right), \tag{6.8}$$

a relation expanding relation (6.7).

$$P\left(\bigcap_{k=1}^{n}A_k\right)=P(A_1)\dots P(A_n) \tag{6.9}$$

is true in the case of independence of the n considered events. If the event B is fixed and of strictly positive probability, relation (6.4) gives the way to define a *new* probability measure on $(\Omega,\Im)$ denoted P_B as follows:

$$P_B(A)=\frac{P(A\cap B)}{P(B)}, \forall A\in\Im. \tag{6.10}$$

P_B is in fact a probability measure as it is easy to verify that it satisfies conditions (2.10) and (2.11) and so P_B is called the *conditional probability measure given B*. The integral with respect to this measure is called the conditional expectation E_B relative to P_B. From relation (6.10) and since $P_B(B)=1$, we thus obtain for any integrable r.v. Y:

$$E_B(Y)=\int_\Omega Y(\omega)dP_B=\frac{1}{P(B)}\int_\Omega Y(\omega)dP\,. \tag{6.11}$$

For our next step, we shall now consider a countable event partition ($B_n, n\geq 1$) of the sample space Ω. That is,

$$\Omega=\bigcup_{n=1}^{\infty}B_n,\; B_i\cap B_j=\varnothing,\; \forall i,j: i\neq j\,. \tag{6.12}$$

Then, for every event A, we have

$$P(A)=\sum_{n\geq 1}P(B_n\cap A) \tag{6.13}$$

and by relation (6.10),

$$P(A)=\sum_{n\geq 1}P(B_n)P\left(A\middle|B_n\right). \tag{6.14}$$

Now, for any integrable r.v. Y, we can write

$$E(Y)=\sum_{n\geq 1}\int_{B_n}Y(\omega)dP \tag{6.15}$$

and from relation (6.11),

$$E(Y)=\sum_{n\geq 1}P(B_n)E_{B_n}(Y)\,. \tag{6.16}$$

As the partition $B_n, n\geq 1$ generates a sub-σ-algebra of $\Im$ denoted $\Im_1$ obtained as the minimal sub-σ-algebra containing all the events of the given partition, we can now define $E_{\Im_1}(Y)$, called *the conditional expectation of Y given* $\Im_1$, as follows:

$$E_{\Im_1}(Y)(\omega)=\sum_{n\geq 1}E_{B_n}(Y)1_{B_n}(\omega)\,. \tag{6.17}$$

It is very important to understand that this conditional expectation is a function of ω and so a new random variable. So, the random variable $E_{\Im_1}(Y)$ assumes on each set B the value of $E_{B_n}(Y)$ that is constant and defined by relation (6.11) with $B=B_n$.

Now, let us compute the expectation of this new random variable $E_{\Im_1}(Y)$; from relation (6.17), we can deduce that

$$\begin{aligned}E\left(E_{\Im_1}(Y)(\omega)\right) &= E\left(\sum_{n\geq 1} E_{B_n}(Y)1_{B_n}(\omega)\right)\\ &= \sum_{n\geq 1} E\left(E_{B_n}(Y)1_{B_n}(\omega)\right)\\ &= \sum_{n\geq 1} E_{B_n}(Y)E\left(1_{B_n}(\omega)\right)\\ &= \sum_{n\geq 1} E_{B_n}(Y)P(B_n)\end{aligned} \tag{6.18}$$

and finally from relation (6.16), we get

$$E(E_{\Im_1}(Y)(\omega)) = E(Y). \tag{6.19}$$

Furthermore, since for any set B belonging to $\Im_1$, B is the union of a certain number of events B_n, finite or at the most denumerable, we obtain by integrating both members of relation (6.17):

$$\begin{aligned}\int_B E_{\Im_1}(Y)(\omega)dP &= \sum_{n\geq 1}\int_B E_{B_n}(Y)1_{B_n}(\omega)dP\\ &= \sum_{n\geq 1} E_{B_n}(Y)\int_B 1_{B_n}(\omega)dP\\ &= \sum_{n\geq 1} E_{B_n}(Y)P(B\cap B_n).\end{aligned} \tag{6.20}$$

Using now relation (6.11), we get:

$$\begin{aligned}\int_B E_{\Im_1}(Y)(\omega)dP &= \left(\sum_{n\geq 1}\frac{1}{P(B\cap B_n)}\int_{B\cap B_n} Y(\omega)dP\right)P(B\cap B_n)\\ &= \sum_{n\geq 1}\int_{B\cap B_n} Y(\omega)dP\\ &= \int_B Y(\omega)dP.\end{aligned} \tag{6.21}$$

In conclusion, we get

$$\int_B E_{\Im_1}(Y)(\omega)dP = \int_B Y(\omega)dP, B\in\Im_1. \tag{6.22}$$

Of course, for $B = \Omega$, this last relation is identical to (6.19).

We shall focus our attention on the meaning of result (6.22) which equates two integrals on every set B belonging to $\Im_1$ but presenting an essential difference: in the left member, the integrand $E_{\Im_1}(Y)$ is $\Im_1$-measurable but in the right member, the integrand Y is $\Im$-measurable and so not necessarily $\Im_1$-measurable since $\Im_1 \subset \Im$. Furthermore, the function $E_{\Im_1}(Y)$ is a.s. unique; indeed, let us suppose that there exists another function $\Im_1$-measurable f so that relation (6.22) is still true. Consequently, we have

$$\int_B f(\omega)dP = \int_B Y(\omega)dP, B \in \Im_1. \quad (6.23)$$

From relations (6.22) and (6.23), we obtain

$$\int_B f(\omega)dP = \int_B E_{\Im_1}(\omega)dP, B \in \Im_1 \quad (6.24)$$

so that

$$\int_B f(\omega)dP = \int_B (f(\omega) - E_{\Im_1}(\omega))dP, B \in \Im_1 . \quad (6.25)$$

As this relation holds for all B belonging to $\Im_1$, it follows that $E_{\Im_1}(Y) = f$, a.s.; otherwise, there would exist a set B belonging to $\Im_1$ so that the difference $E_{\Im_1}(Y) - f$ would be different from 0 and so also the integral

$$\int_B (f(\omega) - E_{\Im_1}(\omega))dP \quad (6.26)$$

in contradiction with property (6.25).

6.2 Conditioning: General Case

We can now extend the definition (6.17) to *arbitrary* sub-σ-algebras using property (6.22) as a definition with the help of the Radon-Nikodym theorem, Halmos (1974).

Definition 6.1 *If* $\Im_1$ *is a sub-*σ*-algebra of* $\Im$*, the conditional expectation of the integrable r.v. Y given* $\Im_1$*, denoted by* $E_{\Im_1}(Y)$ *or* $E\left(Y \middle| \Im_1\right)$*, is any one r.v. of the equivalence class such that:*

(i) $E_{\Im_1}(Y)$ is $\Im_1$*-measurable,*

(ii) $$\int_B E_{\Im_1}(Y)(\omega)dP = \int_B Y(\omega)dP, B \in \Im_1. \quad (6.27)$$

In fact, the class of equivalence contains all the random variables a.s. equally satisfying relation (6.27).

Remark 6.1 Taking $B = \Omega$ in relation (6.27), we get

$$E(E_{\Im_1}Y)) = E(Y), \quad (6.28)$$

a relation extending relation (6.17) to the general case.

Particular cases

(i) $\Im_1$ *is generated by one r.v.* X.

This case means that $\Im_1$ is the sub-σ-algebra of $\Im$ generated by all the inverse images of X and we will use as notation

$$E_{\Im_1}(Y) = E\left(Y \middle| X\right), \tag{6.29}$$

and this conditional expectation is called the *conditional expectation of Y given X.*

(ii) $\Im_1$ *is generated by* n *r.v.* $X_1,...,X_n$.

This case means that $\Im_1$ is the sub-σ-algebra of $\Im$ generated by all the inverse images of $X_1,...,X_n$ and we will use as notation

$$E_{\Im_1}(Y) = E(Y \mid X_1,...,X_n), \tag{6.30}$$

and this conditional expectation is called the *conditional expectation of Y given* $X_1,...,X_n$. In this latter case, it can be shown (Loeve (1977)) that there exists a version $\varphi(X_1,...,X_n)$ of the conditional expectation so that φ is a Borel function from $\mathbb{R}^n$ to $\mathbb{R}$ and as such it follows that $E(Y \mid X_1,...,X_n)$ is constant on each set belonging to $\Im_1$ for which $X_1(\omega) = x_1,...,X_n(\omega) = x_n$, for instance. This justifies the abuse of notation

$$E\left(Y \middle| X_1(\omega) = x_1,...,X_n(\omega) = x_n\right) = \varphi(x_1,...,x_n) \tag{6.31}$$

representing the value of this conditional expectation on all the ω's belonging to the set $\left\{\omega : X_1(\omega) = x_1,...,X_n(\omega) = x_n\right\}$. Taking $B = \Omega$ in relation (6.28), we get

$$E(Y) = \int_{R_n} E\left(Y \middle| X_1(\omega) = x_1,...,X_n(\omega) = x_n\right) dP(X_1(\omega) \le x_1,...,X_n(\omega) \le x_n), \tag{6.32}$$

a result often used in the sequel to evaluate the mean of a random variable using its conditional expectation with respect to some given event.

(iii) If $\Im_1 = \left\{\varnothing, \Omega\right\}$, we get $E(Y \mid \Im_1) = E(Y)$ and if $\Im_1 = \left\{\varnothing, B, B^c, \Omega\right\}$, then $E(Y \mid \Im_1) = E(Y \mid B)$ on B and $E(Y \mid \Im_1) = E(Y \mid B^c)$ on B^c.

(iv) Taking as r.v. Y the indicator of the event A, that is to say

$$1_{A(\omega)} = \begin{cases} 1, \omega \in A, \\ 0, \omega \notin A, \end{cases} \tag{6.33}$$

the conditional expectation becomes the *conditional probability of A given* $\Im_1$ denoted as follows:

$$P(A \mid \Im_1) = E(1_A(\omega) \mid \Im_1) \tag{6.34}$$

and then relation (6.27) becomes

$$\int_B P\left(A \middle| \Im_1(\omega)\right) dP = P(A \cap B), B \in \Im_1. \tag{6.35}$$

Letting $B = \Omega$ in this final relation, we get

$$E\left(P\left(A \middle| \Im_1\right)\right) = P(A), \tag{6.36}$$

a property extending the *theorem of total probability* (6.14). If moreover, A is independent of $\Im_1$, that is to say, if for all B belonging to $\Im_1$

$$P(A \cap B) = P(A)P(B), \tag{6.37}$$

then we see from relation (6.34) that

$$P\left(A \middle| \Im_1\right)(\omega) = P(A), \omega \in \Omega. \tag{6.38}$$

Similarly, if the r.v. Y is independent of $\Im_1$, that is to say if for each event B belonging to $\Im_1$ and each set A belonging to the σ-algebra generated by the inverse images of Y, denoted by $\sigma(Y)$, the relation (6.37) is true, then from relation (6.27), we have

$$E\left(Y \middle| \Im_1\right) = E(Y). \tag{6.39}$$

Indeed, from relation (6.37), we can write that

$$\begin{aligned}\int_B E_{\Im_1}(Y)(\omega)dP &= \int_B Y(\omega)dP, B \in \Im_1 \\ &= E\left(Y1_B\right) \\ &= E(Y)P(B) \\ &= \int_B E(Y)dP,\end{aligned} \tag{6.40}$$

and so, relation (6.39) is proved. In particular, if $\Im_1$ is generated by the r.v. $X_1, \ldots, X_n$, then the independence between Y and $\Im_1$ implies, that

$$E\left(Y \middle| X_1, \ldots, X_n\right) = E(Y). \tag{6.41}$$

Relations (6.39) and (6.41) allow us to have a better understanding of the *intuitive meaning of conditioning*. Under independence assumptions, conditioning has absolutely no impact, for example, on the expectation or the probability, and on the contrary, dependence implies that the results with or without conditioning will be different, this fact meaning that we can interpret conditioning as given *additional information* useful to get more precise results in the case of dependence of course.

The properties of expectation, quoted in section 4, are also properties of conditional expectation, true now a.s., but there are supplementary properties which are very important in stochastic modelling. They are given in the next proposition.

Proposition 6.1 (*Supplementary properties of conditional expectation*)
On the probability space $(\Omega, \Im, P)$, *we have the following properties:*

(i) *If the r.v. X is $\Im_1$-measurable, then*

$$E(X \mid \Im_1) = X, a.s. \tag{6.42}$$

(ii) *Let X be a r.v. and Y $\Im_1$-measurable, then*

$$E(XY|\Im_1) = YE(X|\Im_1), a.s. \tag{6.43}$$

This property means in fact that $\Im_1$*-measurable random variables are like constants for the classical expectation.*

(iii) *Since from relation (6.27), we have that* $E_{\Im}(Y) = Y$, *taking* $Y = E_{\Im_1}(Y)$, *we see that*

$$E_{\Im}(E_{\Im_1}(Y)) = E_{\Im_1}(Y), \tag{6.44}$$

and of course since

$$E_{\Im_1}(E_{\Im}(Y)) = E_{\Im_1}(Y), \tag{6.45}$$

putting these last two relations together, we get

$$E_{\Im}(E_{\Im_1}(Y)) = E_{\Im_1}E_{\Im}((Y)) = E_{\Im_1}(Y). \tag{6.46}$$

This last result may be generalised as follows.

Proposition 6.2 (*Smoothing property of conditional expectation*) *Let* $\Im_1$, $\Im_2$ *be two sub-*σ*-algebras of* $\Im$ *such that* $\Im_1 \subset \Im_2$; *then it is true that*

$$E_{\Im_2}(E_{\Im_1}(Y)) = E_{\Im_1}(E_{\Im_2}(Y)) = E_{\Im_1}(Y), \tag{6.47}$$

a property called the smoothing property in Loeve (1977).

A particular case of relation (6.47) is for example that

$$E\left(E\left(Y|X_1,...,X_n\right)|X_1\right) = E\left(E\left(Y|X_1\right)|X_1,...,X_n\right) = E\left(Y|X_1\right). \tag{6.48}$$

This type of property is very useful for computing probabilities using conditioning and will often be used in the following chapters.

Here is an example illustrating this interest for sums of a random number of random variables with the so-called *Wald identities*.

Example 6.1 (*Wald's identities*) Let $(X_n, n \geq 1)$ be a sequence of i.i.d. real random variables and N a non-negative r.v. with integer values independent of the given sequence. The random variable defined by

$$S_N = \sum_{n=1}^{N} X_n \tag{6.49}$$

is called a *sum of a random number of random variables* and the problem to be solved is the computation of the mean and the variance of this sum supposing that the r.v. X_n have a variance.

From relation (6.28), we have that

$$E(S_N) = E\left(E\left(S_N|N\right)\right) \tag{6.50}$$

and as, from the independence assumptions,

$$E\left(S_N|N\right) = NE(X), \tag{6.51}$$

we also have

$$E(S_N) = E(N)E(X), \tag{6.52}$$

the so-called the first *Wald's identity*:
For the variance of S_N, as

$$\operatorname{var}(S_N) = E(S_N^2) - (E(S_N))^2, \tag{6.53}$$

it suffices to evaluate $E(S_N^2)$ as we did above. From relation (6.28), we can write that:

$$E(S_N^2) = E\left(E\left(S_N^2 \middle| N\right)\right). \tag{6.54}$$

As

$$E\left(S_N^2 \middle| N\right) = E\left(\left(\sum_{i=1}^{N} X_i\right)^2 \middle| N\right), \tag{6.55}$$

we obtain on the set $\{\omega : N(\omega) = n\}$:

$$\begin{aligned} E\left(S_N^2 \middle| N = n\right) &= E\left(\sum_{i=1}^{n} X_i\right)^2 \\ &= \operatorname{var}\left(\sum_{i=1}^{n} X_i\right) + \left(E\left(\sum_{i=1}^{n} X_i\right)\right)^2 \\ &= n \operatorname{var}(X) + (nE(X))^2. \end{aligned} \tag{6.56}$$

Therefore, from relation (6.54),

$$\begin{aligned} E(S_N^2) &= E(N \operatorname{var}(X) + N^2(E(X)^2)) \\ &= E(N)\operatorname{var}(X) + E(N^2)(E(X)^2), \end{aligned} \tag{6.57}$$

and thus, by relations (6.54) and the first Wald's identity (6.53), we get:

$$\operatorname{var}(S_N) = E(N)\operatorname{var}(X) + E(N^2)(E(X^2)) - (E(X))^2(E(N))^2 \tag{6.58}$$

and finally we obtain the *second Wald's identity* in the form

$$\operatorname{var}(S_N) = E(N)\operatorname{var}(X) + \operatorname{var}(N)(E(X))^2. \tag{6.59}$$

6.3 Regular Conditional Probability

The general definition (6.27) gives the conditional expectation of a r.v. Y given $\Im_1$ by an *implicit* relation. Now the question is: can we define the conditional expectation with an *explicit* relation? To give the answer, let us begin with the conditional probability defined by relation (6.34); We know that the conditional probability $P\left(A \middle| \Im_1\right)$ is a r.v., not unique but a.s. unique and this implies that $P\left(A \middle| \Im_1\right)$ as a set function on $\Im$ has a.s. the properties of probability measures:

(i) $P\left(\Omega \middle| \Im_1\right)(\omega) = 1$, (6.60)

(ii) $P\left(A\middle|\Im_1\right)(\omega)\geq 0\ \forall A\in\Im,$ (6.61)

(iii) $P\left(\bigcup_{n=1}^{\infty}A_n\middle|\Im_1\right)(\omega)=\sum_{n=1}^{\infty}P\left(A_n\middle|\Im_1\right)(\omega),\ A_n\in\Im,\forall n,A_i\cap A_j=\varnothing,$ (6.62)

$\forall i,\forall j,i\neq j.$

It is important to note here that the null events N_1,N_2,N_3, on which respectively these last three properties are not true, are generally not identical, so that for each ω, the random set function $P\left(.\middle|\Im_1\right)(\omega)$ from $\Im$ to $\left[0,1\right]$ is not necessarily a probability measure since, to be so, these three sets must be identical. That is why we must introduce the concept of *regular* conditional probability (see Loeve (1977) or Gikhman and Skorokhod (1980)).

Definition 6.2 *The conditional probability* $P\left(.\middle|\Im_1\right)(\omega)$ *is a regular conditional probability if there exists a function p(.,.) from* $\Im\times\Omega$ *to* $\left[0,1\right]$ *so that:*

(i) *for almost all* ω *of* Ω, $p(.,\omega)$, *as a set function on* $\Im$, *is a probability measure,*

(ii) *for every fixed event A belonging to* $\Im$, *p(A,.) is* $\Im$*-measurable and is a version of the given conditional probability, that is a.s., we have*

$$p(A,\omega)=P\left(A\middle|\Im\right)(\omega). \tag{6.63}$$

The interest of such regular conditional probabilities is that we can express the related conditional expectation of an integrable r.v. X a.s. as an integral with respect to the measure $p(.,\omega)$:

$$E\left(X\middle|\Im_1\right)(\omega)=\int_{\Omega}X(\omega')p(d\omega',\omega). \tag{6.64}$$

In many applications, it is sufficient to restrict the attention to all events of the sub-σ-algebra generated by a r.v. X, with values in the measurable space (E,ψ), and denoted by $\sigma(X)$. This means that we are only interested in the following conditional probabilities:

$$P\left(A\middle|\Im_1\right),\ A\in\sigma(X). \tag{6.65}$$

If the conditional probability given $\Im_1$ is regular, we can then define the function C from $\sigma(X)\times\Omega$ to $\left[0,1\right]$ as

$$C(A,\omega)=P\left(A\middle|\Im_1\right)(\omega),A\in\sigma(X),\omega\in\Omega \tag{6.66}$$

satisfying

(i) for almost all ω of Ω, $C(.,\omega)$, as a set function on $\sigma(X)$, is a probability measure,

(ii) for every fixed event A belonging to $\sigma(X)$, $C(A,.)$ is $\Im$-measurable,

(iii) for every event A belonging to $\sigma(X)$ and for every event B belonging to $\Im$, we have

$$\int_B C(A,\omega)P(d\omega) = P(A \cap B)\,. \tag{6.67}$$

C is called the *conditional distribution of X given* $\Im_1$ and the *mixed conditional distribution of X given* $\Im_1$ is defined as the function $Q(.,.)$ from $\psi \times \Omega$ to $[0,1]$ defined by:

$$Q(S,\omega) = P\left(\left\{\omega' : X(\omega') \in S \middle| \Im_1\right\}\right)(\omega), S \in \psi\,. \tag{6.68}$$

The problem of the existence of regular conditional probability was solved by Loeve (1977) or Gikhman and Skorokhod (1980). For our goal, let us just say that this is the case if $\Im$ is generated by a finite or countable family of random variables or if the space E is a complete separable metric space. In the particular case of an n-dimensional real r.v. $X=(X_1,...,X_n)$,we can now introduce the very useful definition of the *conditional distribution function of X given* $\Im_1$ defined as follows:

$$\begin{aligned} F_{\Im_1}(x_1,...,x_n,\omega) &= P\left(X_1 \le x_1,...,X_n \le x_n \middle| \Im_1\right) \\ &= Q\left(\left\{\omega' : X_1(\omega') \le x_1,...,X_n(\omega') \le x_n\right\},\omega\right). \end{aligned} \tag{6.69}$$

Another useful definition concerns an extension of the concept of the independence of random variables to the definition of *conditional independence of the n variables* $X_1,\ldots,X_n$. For all $(x_1,\ldots,x_n)$ belonging to $\mathbb{R}^n$, we have the following identity:

$$F_{\Im_1}(x_1,...,x_n,\omega) = \prod_{k=1}^{n} F_{\Im}(x_k,\omega), \tag{6.70}$$

where of course we have

$$F_{\Im}(x_k,\omega) = P\left(X_k \le x_k \middle| \Im_1\right) \tag{6.71}$$

according to the definition (6.69) with n=1.

Example 6.2 On the probability space $(\Omega,\Im,P)$, let (X,Y) be a two-dimensional real r.v. whose d.f. is given by

$$F(x,y) = P(X \le x, Y \le y). \tag{6.72}$$

As $\mathbb{R}^2$ is a complete separable metric space, it is said above that there exist regular conditional probabilities given the sub-σ-algebras $\sigma(X)$ or $\sigma(Y)$ and so the related conditional d.f. denoted by

$$F_{X|Y}\left(x\middle|\{\omega : Y = y\}\right),\; F_{Y|X}\left(y\middle|\{\omega : X = x\}\right) \tag{6.73}$$

also exists. If moreover, the d.f. F has a density f, we can also introduce the concept of *conditional density* for both functions $F_{X|Y}$, $F_{Y|X}$, F_X, giving at the same time an intuitive interpretation of conditioning in this special case. We know that for every fixed (*x,y*):

$$f(x,y)\Delta x\Delta y+o(x,y,\Delta x,\Delta y)=P(x<X\le x+\Delta x,y<Y\le y+\Delta y), \quad (6.74)$$

where $o(x,y,\Delta x,\Delta y)\to 0$ for $(\Delta x,\Delta y)\to(0,0)$ and similarly for the marginal density function of *X*:

$$f_X(x)\Delta x+\bar{o}(x,\Delta x)=P(x<X\le x+\Delta x), \quad (6.75)$$

where $\bar{o}(x,\Delta x)\to 0$ for $\Delta x\to 0$ with of course:

$$f_X(x)=\int_R f(x,y)dy\,. \quad (6.76)$$

Using formula (6.4), we thus obtain

$$P\left(y<Y\le y+\Delta y\middle|x<X\le x+\Delta x\right)=\frac{f(x,y)\Delta x\Delta y+o(x,y,\Delta x,\Delta y)}{f_X(x)\Delta x+\bar{o}(x,\Delta x)}. \quad (6.77)$$

Letting Δx tend to 0, we get

$$\lim_{\Delta x\to 0}P\left(y<Y\le y+\Delta y\middle|x<X\le x+\Delta x\right)=\frac{f(x,y)}{f_X(x)}\Delta y\,. \quad (6.78)$$

This relation shows that the function $f_{Y|X}$ defined by

$$f_{Y|X}\left(y\middle|x\right)=\frac{f(x,y)}{f_X(x)} \quad (6.79)$$

is the *conditional density of Y, given X*. Similarly the *conditional density of X, given Y,* is given by

$$f_{X|Y}\left(x\middle|y\right)=\frac{f(x,y)}{f_Y(y)}\,. \quad (6.80)$$

Consequently, for any Borel subsets *A* and *B* of $\mathbb{R}$, we have

$$\begin{aligned}&P\left(X\in A\middle|Y(\omega)=y\right)=\int_A f_{X|Y}\left(x\middle|y\right)dx=\frac{1}{f_Y(y)}\int_A f(x,y)dx,\\&P((X,Y)\in A\cap B)=\int_{A\cap B}f(x,y)dxdy=\int_B\left(\int_A f_{X|Y}\left(x\middle|y\right)dx\right)f_Y(y)dy.\end{aligned} \quad (6.81)$$

The last equalities show that the density of (*X,Y*) can also be characterised by one marginal d.f. and the associated conditional density, as from relations (6.78) and (6.81):

$$f=f_X\times f_{Y|X}=f_Y\times f_{X|Y}\,. \quad (6.82)$$

It is possible that *conditional means* exist; if so they are given by the relations

$$E\left(X\middle|Y=y\right)=\int_{\mathbb{R}}f\left(x\middle|y\right)dx,\; E\left(Y\middle|X=x\right)=\int_{\mathbb{R}}f\left(y\middle|x\right)dy\,. \quad (6.83)$$

The conditional mean of *X* (resp.*Y*) given *Y*=*y* (resp. *X*=*x*)can be seen as a function of the real variable *y* (resp. *x*) called *the regression curve of X* (*resp.Y*) *given Y* (*resp. X*). The two regression curves will generally not coincide and not be straight lines except if the two r.v. X and Y are independent because, in this case, we have from relations (6.78) and (6.80) that

$$f_{X|Y} = f_X, f_{Y|X} = f_Y \tag{6.84}$$

and so:

$$E(X|Y) = E(X), E(Y|X) = E(Y), \tag{6.85}$$

proving thus that the two regression curves are straight lines parallel to the axes passing through the "centre of gravity" ($E(X)$, $E(Y)$) of the probability mass in $\mathbb{R}^2$.

In the special case of a non-degenerated normal distribution for (X,Y) with vector mean (m_1,m_2) and variance covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}, \tag{6.86}$$

it can be shown that the two conditional distributions are also normal with parameters

$$\begin{aligned} Y|X &\prec N_2\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), \sigma_2^2(1-\rho^2)\right), \\ X|Y &\prec N_2\left(\mu_1 + \frac{1}{\rho}\frac{\sigma_1}{\sigma_2}(y-\mu_2), \sigma_2^2(1-\rho^2)\right). \end{aligned} \tag{6.87}$$

Thus, the two regression curves are linear.

7 STOCHASTIC PROCESSES

In this section, we shall always consider a *complete* probability space $(\Omega, \Im, \mathrm{P})$ with a *filtration F*. Let us recall that a probability space $(\Omega, \Im, \mathrm{P})$ is *complete* if every subset of an event of probability 0 is measurable, i.e., in the σ-algebra $\Im$, and so also of probability 0.

Definition 7.1 *F being a filtration on the considered basic probability space means that F is a family* $(\Im_t, t \in T)$ *of sub-*σ*-algebras of* $\Im$*, the index set T being either the natural set* $\{0,1,...,n,...\}$ *or the positive real half-line* $[0,\infty)$*, such that:*

$$\begin{aligned} &\text{(i)} \quad s < t \Rightarrow \Im_s \subset \Im_t, \\ &\text{(ii)} \quad \Im_t = \bigcap_{u>t} \Im_u, \end{aligned} \tag{7.1}$$

(iii) $\Im_0$ contains all subsets with probability 0.

Assumption (ii) is called the *right continuity property* of the filtration *F*. Any filtration satisfying these three assumptions is called a filtration *satisfying the* *usual assumptions*. The concept of filtration can be interpreted as a family of amounts of information so that $\Im_t$ gives all the observable events at time *t*.

Definition 7.2 *The quadruplet* $\left(\left(\Omega, \Im, \mathrm{P}, \left(\Im_t, t \in T\right)\right)\right.$ *is called a filtered probability space.*

Definition 7.3 *A random variable* $\tau : \Omega \mapsto T$ *is a stopping time i:*

$$\forall t \in T: \ \left\{\omega : \tau(\omega) \leq t\right\} \in \Im_t. \tag{7.2}$$

The interpretation is the following: the available information at time *t* gives the possibility to observe the event given in (7.2) and to decide for example if one stops the future observations after time *t,* or not. We have the following proposition:

Proposition 7.1 *The random variable* τ *is a stopping time if and only if*

$$\left\{\omega : \tau(\omega) < t\right\} \in \Im_t, \ \forall t \in T. \tag{7.3}$$

Definition 7.4 *A stochastic process (or simply process) with values in the measurable space* $\left(E, \aleph\right)$ *is a family of random variables*

$$\left\{X_t, t \in T\right\} \tag{7.4}$$

where for all t:

$X_t : \Omega \mapsto E, \ \left(\Im, \aleph\right) -$ *measurable.*

This means, in particular, that for every subset *B* of the σ-algebra $\aleph$, the set

$$X_t^{-1}\left(B\right) = \left\{\omega : X_t(\omega) \in B\right\} \tag{7.5}$$

belongs to the σ-algebra $\Im$.

Remark 7.1 If $\left(E, \aleph\right) = \left(\mathbb{R}, \beta\right)$, the process is called a *real stochastic process* with values in $\mathbb{R}$; if $\left(E, \aleph\right) = \left(\mathbb{R}^n, \beta^n\right)$, it is called a *real multidimensional process* with values in $\mathbb{R}^n$.

If T is the natural set $\left\{0, 1, \ldots, n, \ldots\right\}$, the process X is called a *discrete time stochastic process* or a *random sequence*; if *T* is the positive real half-line $\left[0, \infty\right)$, the process *X* is called a *continuous time stochastic process.*

Definition 7.5 *The stochastic process X is adapted to the filtration F if, for all t, the r.v.* X_t *is* $\Im_t$*-measurable. This means that, for all* $t \in T$,

$$X_t^{-1}(B) = \{\omega : X_t(\omega) \in B\} \in \Im_t, \forall B \in \aleph. \tag{7.6}$$

Definition 7.6 *Two processes X and Y are indistinguishable if a.s., for all* $t \in T$,

$$X_t = Y_t. \tag{7.7}$$

This means that

$$\mathrm{P}(X_t = Y_t, \forall t \in T) = 1. \tag{7.8}$$

Definition 7.7 *The process X (or Y) is a modification of the process Y (or X) if a.s., for all* $t \in T$,

$$X_t = Y_t \ a.s. \tag{7.9}$$

This means that

$$\mathrm{P}(X_t = Y_t., \forall t \in T) = 1. \tag{7.10}$$

for all $t \in T$.

Definition 7.8 *For every stochastic process X, the function from T to E,*

$$t \mapsto X_t(\omega) \tag{7.11}$$

defined for each $\omega \in \Omega$, *is called a trajectory or sample path of the process.*

It must be clearly understood that the so-called "modern" study of stochastic processes is concerned with the study of the properties of these trajectories. For example, we can affirm that if two processes *X* and *Y* are indistinguishable, then there exists a set *N* belonging to $\Im$ of probability 0 such that:

$$\forall \omega \notin N : X_t(\omega) = Y_t(\omega), \forall t \in T. \tag{7.12}$$

In other words, for each ω element of the set $\Omega - N$, the two functions $t \mapsto X_t(\omega)$ and $t \mapsto Y_t(\omega)$ are equal. As the basic probability space is complete, the neglected set *N* belongs to $\Im_t$, for all $t \in T$.

Definition 7.9 *A real stochastic process X is càdlàg (continu à droite, limite à gauche) if a.s. the trajectories of X are right continuous and have left limits at every point t.*

Definition 7.10 *If X is a real stochastic process and a set* $\Lambda \in \beta$, *then the r.v. defined b:*

$$T(\omega) = \inf\{t > 0 : X_t(\omega) \in \Lambda\} \tag{7.13}$$

is called the hitting time of Λ *by the process X.*

It is easily shown that the **properties** of stopping and hitting times are (see Protter (1990)):
(i) If X is càdlàg, adapted and $\Lambda \in \beta$, then the hitting time related to Λ is a stopping time.
(ii) Let S and T be two stopping times, then the r.v.

$$S \wedge T\left(=\min\{S,T\}\right),\ S \vee T\left(=\max\{S,T\}\right),\ S+T,\ \alpha S(\alpha > 1) \tag{7.14}$$

are also stopping times.

Definition 7.11 *If T is a stopping time, the σ-algebra $\Im_T$ defined by*

$$\Im_T = \left\{\Lambda \in \Im : \Lambda \bigcap \{\omega : T(\omega) \le t\} \in \Im_t, \forall t \ge 0\right\} \tag{7.15}$$

is called the stopping time σ-algebra.

In fact, the σ-algebra $\Im_T$ represents the information of all observable sets up to the stopping time T. We can also say that $\Im_T$ is the smallest stopping time containing all the events related to the r.v. $X_{T(\omega)}(\omega)$ for all the adapted càdlàg processes X or generated by these r.v. We also have for two stopping times S and T,

(i) $S \le T$ a.s. $\Rightarrow \Im_S \subset \Im_T$, (7.16)

(ii) $\Im_S \bigcap \Im_T, = \Im_{S\wedge T}$. (7.17)

8 MARTINGALES

In this section, we shall briefly present some topics related to the most well-known category of stochastic processes called *martingales*. Let X be a real stochastic process defined on the filtered complete probability space $\left(\Omega, \Im, P, \left(\Im_t, t \in T\right)\right)$.

Definition 8.1 *The process X is called a martingale if:*

(i) $\forall t \ge 0, \exists E\left(X_t\right)$, (8.1)

(ii) $s < t \Rightarrow E\left(X_t \mid X_s\right) = X_s$, a.s. (8.2)

The latter equality is called the *martingale property* or the *martingale equality*.

Definition 8.2 *The process X is called a super-martingale (resp. sub-martingale) if:*

(i) $\forall t \ge 0, \exists E\left(X_t\right)$, (8.3)

$$\text{(ii)}\, s < t \Rightarrow E\left(X_t \mid \Im_s\right) \le (\ge) X_s, \text{a.s.} \tag{8.4}$$

The martingale concept is interesting; indeed, as the best estimator at time s ($s>t$) for the value of X_t, as given by the conditional expectation appearing in relation (8.2), the martingale equality means that *the best predicted value* simply is the observed value of the process at the time of predicting s. The use of martingales in finance is frequently (see Janssen and Skiadas (1995)) to model the concept of an *efficient financial market.*

Definition 8.3 *The martingale X is closed if:*

$$\exists\, Y :$$

$$\text{(i)}\mathrm{E}\left(\left|Y\right|\right) < \infty, \tag{8.5}$$

$$\text{(ii)}\forall t \in \left[0,\infty\right) : \mathrm{E}\left(Y \middle| \Im_t\right) = X_t, \text{a.s.}$$

It is possible to prove the following result (see for example Protter (1990)).

Proposition 8.1 (i) *If X is a supermartingale, then the function* $t \mapsto E\left(X_t\right)$ *is right continuous iff there exists a unique modification Y of X such that Y is càdlàg.*
(ii) *If X is a martingale then, up to a modification, the function* $t \mapsto E\left(X_t\right)$ *is right continuous.*

It follows that every martingale such that the function $t \mapsto E\left(X_t\right)$ is right continuous is càdlàg.
The two most important results about martingales are *the martingale convergence theorem* and the *optional sampling (or Doob's) theorem.* Before giving these results, we still need a final technical definition.

Definition 8.4 (*Meyer (1966)*) *A family* $\left(\xi_u, u \in A\right)$ *where A is an infinite index set, is uniformly integrable if:*

$$\lim_{n\to\infty} \sup_{\alpha} \int_{\{\omega : |\xi_\alpha(\omega)| \ge n\}} \left|\xi_\alpha(\omega)\right| d\mathrm{P}(\omega) = 0. \tag{8.6}$$

Proposition 8.2 *Let X be a supermartingale in such a way that the function* $t \mapsto E\left(X_t\right)$ *is right continuous such that:*

$$\sup_{t\in[0,\infty)} E\left(\left|X_t\right|\right) < \infty\,; \tag{8.7}$$

then, there exists a r.v. Y such that

$$\begin{aligned}&(\text{i})E\left(\left|Y\right|\right),\\&(\text{ii})Y=\lim_{t\to\infty}X_t,\ \text{ a.s.}\end{aligned}\tag{8.8}$$

Moreover, if X is a martingale closed by the r.v. Z, then the r.v. Y also closes X and

$$Y=E\left(Z\,\middle|\,\Im_\infty\right),\tag{8.9}$$

where

$$\Im_\infty=\sigma\left(\bigcup_{0\le t<\infty}\Im_t\right).\tag{8.10}$$

With the aid of the concept of uniform integrability, we can obtain the following corollary.

Corollary 8.1 (i) *Let X be a right continuous martingale and uniformly integrable; then the limit*

$$Y=\lim_{t\to\infty}X_t\tag{8.11}$$

exists a.s.; moreover $Y\in L^1$ *and the r.v. Y closes the martingale X.*

(ii) *Let X be a right continuous martingale; then* $X=(X_t,t\ge 0)$ *is uniformly integrable if and only if*

$$Y=\lim_{t\to\infty}X_t\tag{8.12}$$

exists a.s., $Y\in L^1$*, and* $\left(X_t,t\in\left[0,\infty\right]\right)$ *is a martingale with, a.s.*

$$X_\infty=Y.\tag{8.13}$$

Now, an interesting question is: what happens if we observe a martingale *X* at two stopping times *S,T* (*S*<*T*, a.s.)? The reply is given by the so-called *optional sampling theorem* also called *Doob's theorem.*

Proposition 8.3 (*The optional sampling theorem or Doob's theorem*) *Let X be a right continuous martingale closed by* X_∞ *and let S and T be two stopping times so that a.s. S < T; then the r.v.* $X_S,X_T\in L^1$ *and*

$$X_S=E\left(X_T\,\middle|\,\Im_S\right),\ \text{a.s.}\tag{8.14}$$

This important theorem means that if we restrict the random observation time set to $\{S,T\}$, then the restriction of the martingale to this set is *still* a martingale provided that *S* and *T* are two stopping times with of course *S*<*T*, a.s. This result is interesting for the concept of *stopped process.*

Definition 8.5 *Let X be a stochastic process and T a stopping time. The stopped stochastic process* X^T *is defined by*

$$X^T = \left(X_t^T, t \in [0, \infty]\right) \tag{8.15}$$

where

$$\begin{aligned} &X_t^T(\omega) = X_{t \wedge T}(\omega), \\ &\textit{with } t \wedge T = \inf\{t, T\}. \end{aligned} \tag{8.16}$$

From this definition, it follows that if the process X is adapted and càdlàg, then so is the stopped process X^T. This is due to the fact that $t \wedge T$ is also a stopping time and moreover,

$$X_t^T = X_t 1_{\{t<T\}} + X_T 1_{\{t \geq T\}}. \tag{8.17}$$

This leads to the last result we want to mention.

Proposition 8.4 *Let X be a right continuous uniformly integrable martingale; then the stopped process* $X^T = \left(X_{t \wedge T}, t \in [0, \infty]\right)$ *has the same properties with respect to the filtration* $\left(\Im_t, t \in [0, \infty]\right)$.

9 BROWNIAN MOTION

There are a lot of particular stochastic processes and some of them will be extensively studied in the sequel, such as renewal processes, random walks, Markov chains, semi-Markov and Markov processes and their main extensions. However, to finish this introduction to probability theory, we want to introduce briefly the concept of *Brownian motion* or *Brownian process*. We will work on a basic complete filtered probability space satisfying the usual assumptions and denoted $\left(\Omega, \Im, P, \left(\Im_t, t \in [0, \infty)\right)\right)$.

Definition 9.1 *The real stochastic process* $B = \left(B_t, t \in [0, \infty)\right)$ *will be called a Brownian motion or Brownian or Wiener process with trend* μ *and variance* σ^2 *provided that:*

(i) *B is adapted to the basic filtration,*

(ii) *B has independent increments,*

i. e. that:

$$\begin{aligned} &\forall s, t \ (0 \leq s < t) \ : \ P\left(B_t - B_s \in A \middle| \Im_S\right) = P\left(B_t - B_s \in A\right), \\ &\forall \textit{Borel sets } B, \end{aligned} \tag{9.1}$$

(iii) *B* has *stationary increments*, *i.e.*:

$\forall s,t \ (0 \le s < t)$: $B_t - B_s$ *has a normal distribution* $N(\mu(t-s), \sigma^2(t-s))$, (9.2)

(iv) $P(B_0 = x) = 1, \ (x \in \mathbb{R})$. (9.3)

If moreover, we have

$$\mu = 0, \ \sigma^2 = 1, \ x = 0, \tag{9.4}$$

then the Brownian motion is called *standard*.
Let us now give the most important properties of the standard Brownian motion.

Property 9.1 *If B is a Brownian motion, then there exists a modification of B, the process B*, such that B* has, a.s., continuous trajectories.*

Property 9.2 *If B is a standard Brownian motion, then B is a martingale.*

Property 9.3 *If B is a standard Brownian motion, then the process Q where*

$$Q = \left(B_t^2 - t, t \in [0, \infty)\right) \tag{9.5}$$

is a martingale.

Remark 9.1 *It can also be proved that both Properties 9.2 and 9.3 characterise a standard Brownian motion.*

Property 9.4 *If B is a standard Brownian motion, then for almost all* ω*, the trajectory* $\omega \mapsto B_t(\omega)$ *is not of bounded variation on every closed interval* $[a,b]$.

This explains why it is necessary for models in finance and in insurance to define a new type of integral, called the *Itô* or *stochastic integral*, if we want to intregrate with respect to *B* (see for example Protter (1990)). This will be done in Chapter 5, section 4.2.

Chapter 2

RENEWAL THEORY AND MARKOV CHAINS

In this chapter, the reader will find a summary of the basic results on renewal theory and Markov processes useful for understanding of the following chapters. A more detailed version with proofs can be found in Janssen and Manca (2006) (Chapters 2 and 3)

1 PURPOSE OF RENEWAL THEORY

Let us consider the following reliability problem: at time 0, the given system starts with a new component which fails at random time T_1. At this time, a new component immediately enters the system to replace the first one and fails at time *c*. There is another immediate replacement by a new component inserted in the system, still of the same type, and so on.
We will note $(T_n, \quad n \geq 0)$ the successive replacement times, setting, of course:

$$T_0 = 0 . \tag{1.1}$$

The *lifetimes* of the successive components entering the system are given by

$$X_n = T_n - T_{n-1}, \qquad n \geq 1 . \tag{1.2}$$

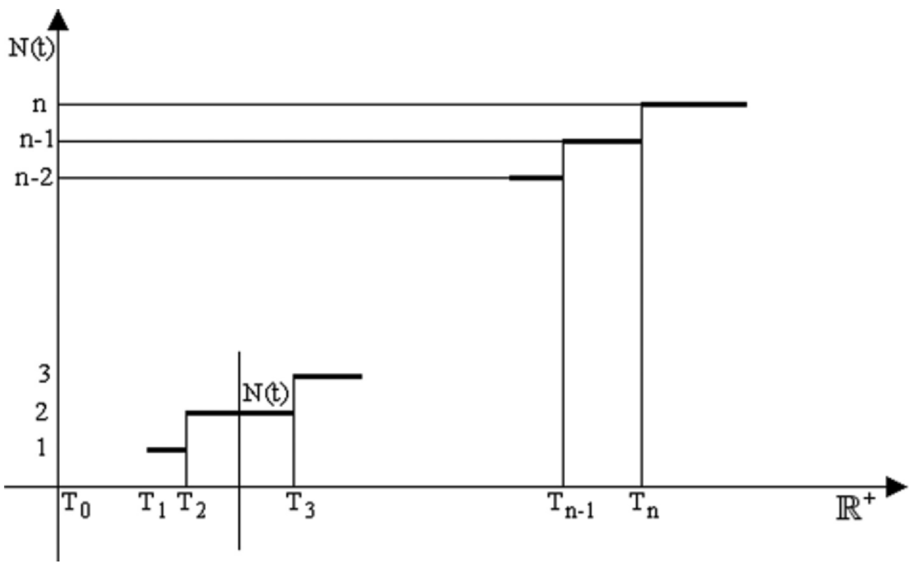

Figure 1.1:trajectory of $N(t)$

From an operational point of view, an important characteristic of the considered system at time t is the total number of replacements occurring on the interval $[0,t]$. Let us remark that, for the moment, we do not take into account the initial component. If $N(t)$ represents the random variable we have just defined, we have, for $n \geq 1$:

$$N(t) > n-1 \Leftrightarrow T_n \leq t. \tag{1.3}$$

It is possible to represent a realization of the stochastic process $(N(t),\quad t \geq 0)$, as shown in **Figure 1.1**.

The first moment of $N(t)$ will give the mean number of replacements on $(0,t]$. In particular, if at time 0 the manager must be able to have an inventory large enough to perform all replacements, the level of the inventory will be, on average, the expectation $E(N(t))$. Of course, the manager must add *buffer stock* to prevent random evolution. This problem will be solved in section 7. The area of probability studying such processes is called *Renewal Theory*. It is, at least for applied probability, one of the most important topics encountered in real life problems.

2 MAIN DEFINITIONS

Let $(X_n,\quad n \geq 1)$ be a sequence of non-negative, independent and identically distributed random variables defined on the probability space $(\Omega, \Im, P)$.

Definition 2.1 *The random sequence* $(T_n, n \geq 0)$*, where:*

$$T_0 = 0 \quad a.s., \tag{2.1}$$

$$T_n = X_1 + \cdots + X_n, \qquad n \geq 1, \tag{2.2}$$

is called a renewal sequence or renewal process.

The r.v. $T_n, n \geq 0$ *are called renewal times and the r.v.* $X_n, n \geq 1$ *are called interarrival times.*

Example 2.1

1) In the first section, we give an example in *reliability theory*.

2) Another important example is *queueing theory*. Let us consider a queueing system composed of a server, a process of customer arrivals, a process of service times and a discipline rule of the type "first in, first out" (FIFO), which means the first customer present in the system is the first served.

In many models of queueing theory, the arrival process is assumed to be a renewal process.

In this case, the r.v. T_n is the arrival time of the *n*th customer, assuming that a customer number 0 is immediately served at time 0, and the r.v. X_n represents the interarrival time between the ($n-1$)th and the *n*th customer.

3) An arrival process is also considered in *risk theory*. Let us consider an insurance company starting at time 0 with a capital amount u $(u \geq 0)$ called the initial *reserve*. The customers pay premiums, and the insurance company has to pay for the accidents claimed by the customers. In this case, the r.v. T_n represents the arrival of the *n*th claim, assuming that the company just starts at the arrival of a claim called claim 0, and the r.v. X_n is the interarrival time between the ($n-1$)th and the *n*th claims $(n \geq 1)$.

4) In *counter theory*, we consider particles arriving at times T_n, $n \geq 0$ with $T_0 = 0$ so that here too, the r.v. X_n are interpreted in terms of interarrival time between two successive particles.

Definition 2.2 *With each renewal sequence, we can associate the following stochastic process, continuous in time, with values in* $\mathbb{N}$:

$$(N(t), \quad t \geq 0), \tag{2.3}$$

where

$N(t) > n-1 \Leftrightarrow T_n \leq t, \qquad n \in \mathbb{N}_0$.

This process is called the associated counting process or the renewal counting process.

$N(t)$ *represents the total number of "renewals" on* $(0,t]$.

Definition 2.3 *The renewal function is defined as*

$$H(t) = E(N(t)) \tag{2.4}$$

provided the expectation is finite.

3 CLASSIFICATION OF RENEWAL PROCESSES

Let us suppose that the random variables are defined on $\overline{\mathbb{R}}$ with distribution function *F* such that, to avoid trivialities:

$$F(0) < 1. \tag{3.1}$$

If

$$F(+\infty) = 1, \tag{3.2}$$

we have the usual case of real random variables.

From relation (2.2), we get

$$\mathrm{P}(N(t) > n-1) = F^{(n)}(t), \qquad n \geq 1, \tag{3.3}$$

$F^{(n)}$ being the *n*-fold *convolution product* of *F* with itself.

Since, for $n \geq 1$:

$$P(\,N(t)=n\,)=P(\,N(t)>n-1\,)-P(\,N(t)>n\,)\ ,\tag{3.4}$$

using relation (3.3), we obtain:

$$P(\,N(t)=n\,)=F^{(n)}(t)-F^{(n+1)}(t),\qquad n\geq 1\,.\tag{3.5}$$

$F^{(0)}$ is defined as being the Heaviside distribution with a unit mass at the origin, i.e.,

$$F^{(0)}=U_0\,;\tag{3.6}$$

the relation (3.5) is still valid for $n=0$, since

$$\mathrm{P}(\,N(t)=0\,)=1-F(t).\tag{3.7}$$

Using Stein's lemma (1946), the following very important result can be proved:

Proposition 3.1 *If* $F(0)<1$, *then, for all* t, $N(t)$ *has moments of any order.*

In particular, this proposition implies that the renewal function is finite for all finite *t*. Consequently, we can write successively:

$$\begin{aligned}E(\,N(t)\,)&=\sum_{n=1}^{\infty}n\big[\,F^{(n)}(t)-F^{(n+1)}(t)\,\big]\\&=F(t)-F^{(2)}(t)+2F^{(2)}(t)-2F^{(3)}(t)+\cdots\\&=F(t)+F^{(2)}(t)+F^{(3)}(t)+\cdots,\end{aligned}\tag{3.8}$$

so that using relation (2.4):

$$H(t)=\sum_{n=1}^{\infty}F^{(n)}(t).\tag{3.9}$$

In several cases, it is useful to consider the initial renewal and to define at time *t* the random variable $N'(t)$ as being the total number of renewals on $[\,0,t\,]$. Clearly, we have, for all $t\geq 0$:

$$N'(t)=N(t)+1,\tag{3.10}$$

and consequently:

$$E(\,N'(t)\,)=H(t)+1\,.\tag{3.11}$$

Setting

$$R(t)=E(\,N'(t)\,),\tag{3.12}$$

we get from relations (3.11), (3.9) and (3.6):

$$R(t)=\sum_{n=0}^{\infty}F^{(n)}(t).\tag{3.13}$$

Of course, we have

$$R(t)=U_0(t)+H(t)\,.\tag{3.14}$$

The classification of a renewal process is based on three concepts: *recurrence*, *transience* and *periodicity*.

Definition 3.1
(i) *A renewal process* $(T_n, \quad n \geq 1)$ *is recurrent if* $X_n < \infty$ *for all n; otherwise it is called transient.*
(ii) *A renewal process* $(T_n, \quad n \geq 1)$ *is periodic with period* δ *if the possible values of the random variables* $X_n, \quad n \geq 1$ *form the denumerable set* $\{0, \delta, 2\delta, \ldots\}$, *and* δ *is the largest such number. Otherwise* (*that is, if there is no such strictly positive* δ), *the renewal process is aperiodic.*
A direct consequence of this definition is the characterization of the type of a renewal process, with the help of distribution function *F*.

Proposition 3.2 *A renewal process of distribution function F is*
(i) *recurrent iff* $F(\infty) = 1$,
(ii) *transient iff* $F(\infty) < 1$,
(iii) *periodic with period* δ $(\delta > 0)$ *iff* F *is constant over intervals* $[n\delta, \quad (n+1)\delta), \quad n \in \mathrm{N}$, *and all its jumps occur at points* $n\delta, \quad n \in \mathbb{N}$.

If *t* tends toward $+\infty$, relation (3.9) gives:

$$H(+\infty) = \begin{cases} +\infty & \text{if} \quad F(+\infty) = 1, \\ \dfrac{F(+\infty)}{1 - F(+\infty)} & \text{if} \quad F(+\infty) < 1. \end{cases} \tag{3.15}$$

Or, equivalently by relation (3.13):

$$R(+\infty) = \begin{cases} +\infty & \text{if} \quad F(+\infty) = 1, \\ \dfrac{1}{1 - F(+\infty)} & \text{if} \quad F(+\infty) < 1. \end{cases} \tag{3.16}$$

This proves the next proposition.

Proposition 3.3 *A renewal process of distribution F is recurrent or transient, depending on whether* $H(+\infty) = +\infty$ *or* $H(+\infty) < +\infty$. *In the last case, we have*

$$R(+\infty) = \frac{1}{1 - F(+\infty)} \quad or \quad H(+\infty) = \frac{F(+\infty)}{1 - F(+\infty)}. \tag{3.17}$$

The interest of the classification given above will be clearer with the concept of *lifetime of a renewal process*.

Definition 3.2 *The lifetime of a renewal process* $(T_n, n \geq 1)$ *is the random variable L, possibly defective, defined by:*

$$L = \sup\{T_n : T_n < \infty\}. \tag{3.18}$$

So, if $L = \ell$, this means that there is only a finite number of renewals on $[0,\infty)$. Also, we define a new random variable N giving the total number of renewals on $[0, L)$.

Definition 3.3 *The total number of renewals on* $(0,\infty)$, *possibly infinite, is given by*

$$N = \sup\{N(t),\, t \geq 0\}. \tag{3.19}$$

In reliability theory, the event $\{N = k\}$ would mean that the (k+1)th component introduced in the system would have an infinite lifetime! Also the probability distribution of N is given by

$$P(N = 0) = 1 - F(+\infty), \tag{3.20}$$

$$P(N = 1) = F(+\infty)(1 - F(+\infty)), \tag{3.21}$$

and in general, for $k \in \mathbb{N}$:

$$P(N = k) = (F(+\infty))^k (1 - F(+\infty)). \tag{3.22}$$

Of course if $F(+\infty) = 1$, we have, a.s.,

$$N = +\infty. \tag{3.23}$$

In the case of a transient renewal process, we can write, using relation (3.22):

$$E(N) = \sum_{k=1}^{\infty} k[F(+\infty)]^k (1 - F(+\infty)). \tag{3.24}$$

As the function $\dfrac{1}{1-x}$ can be written for $|x| < 1$ as a power series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \tag{3.25}$$

which is analytical on $(-1,+1)$ and thus derivable, we have

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}. \tag{3.26}$$

Writing relation (3.24) under the form

$$E(N) = F(+\infty)(1 - F(+\infty)).\sum_{k=1}^{\infty} k[F(+\infty)]^{k-1} \tag{3.27}$$

we get, using relation (3.26):

$$E(N) = \frac{F(+\infty)}{1 - F(+\infty)}. \tag{3.28}$$

So, it is possible to compute the mean of the total number of renewals very easily in the transient case. We can also give the distribution function of L. Indeed, we have:

$$P(L \leq t) = \sum_{n=0}^{\infty} P(T_n \leq t, \quad X_{n+1} = +\infty). \tag{3.29}$$

From the independence of T_n and X_{n+1}, we may deduce that:

$$P(L \le t) = 1 - F(+\infty) + \sum_{n=1}^{\infty} F^{(n)}(t)(1 - F(+\infty)). \tag{3.30}$$

And finally, by equality (3.13):

$$P(L \le t) = (1 - F(+\infty))R(t). \tag{3.31}$$

To compute the mean of the lifetime of the process, we use the following trick based on the independence of the random variables X_n, $n \ge 1$; we can successively write:

$$E(L) = E\left(T_1 \cdot I_{\{T_1<\infty\}}\right) + E(L) \cdot E\left(I_{\{T_1<\infty\}}\right), \tag{3.32}$$

$$= F(+\infty)\int_0^{+\infty}\left(1 - \frac{F(t)}{F(+\infty)}\right)dt + E(L) \cdot F(+\infty) \tag{3.33}$$

so that

$$E(L) = \int_0^{+\infty}(F(+\infty) - F(t))dt + E(L) \cdot F(+\infty). \tag{3.34}$$

And finally

$$E(L) = \frac{1}{1 - F(+\infty)}\int_0^{+\infty}(F(+\infty) - F(t))dt. \tag{3.35}$$

So, for a transient renewal process, the lifetime is always a.s. finite and has a finite mean given by relation (3.35).

Example 3.1: *The Poisson process*

In queueing and risk theory presented in **Example 2.1,** the classical assumption for the arrival process is that it constitutes a renewal process where the r.v. X_n has as common distribution function:

$$F(x) = \begin{cases} 0 & , \quad x < 0, \\ 1 - e^{-\lambda x} & , \quad x \ge 0, \end{cases} \tag{3.36}$$

with λ being a fixed positive constant.

As $F(+\infty) = 1$, the arrival process is recurrent. For (3.3), it is possible to have the analytical expression of the successive n-fold convolutions. Indeed, we can successively write:

$$F^{(2)}(t) = \lambda\int_0^t \left(1 - e^{-\lambda(t-x)}\right)e^{-\lambda x}dx \tag{3.37}$$

$$= \lambda\int_0^t \left(e^{-\lambda x} - e^{-\lambda t}\right)dx \tag{3.38}$$

$$= 1 - e^{-\lambda t} - \lambda t e^{-\lambda t} \tag{3.39}$$

$$= 1 - e^{-\lambda t}(1 + \lambda t), \tag{3.40}$$

and in general:

$$F^{(n)}(t) = 1 - e^{-\lambda t}\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}. \tag{3.41}$$

Applying result (3.5), we have

$$\begin{aligned} \mathrm{P}(\,N(t) = n\,) &= 1 - e^{-\lambda t}\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!} - 1 + e^{-\lambda t}\sum_{k=0}^{n}\frac{(\lambda t)^k}{k!} \\ &= e^{-\lambda t}\frac{(\lambda t)^n}{n!}. \end{aligned} \tag{3.42}$$

That is, for all fixed *t*, the process $(\,N(t)\,)$ is a Poisson process of parameter λt. The value of the renewal function *H* follows from relations (3.9) and (3.8):

$$H(t) = \sum_{n=1}^{\infty} n e^{-\lambda t}\frac{(\lambda t)^n}{n!} \tag{3.43}$$

$$= e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{(n-1)!} \tag{3.44}$$

$$= e^{-\lambda t}\lambda t\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}, \tag{3.45}$$

or

$$H(t) = \lambda t. \tag{3.46}$$

It follows that for a Poisson renewal process, the renewal function is linear.
As we shall see in section 5, a renewal process may also be characterized by its renewal function. The Poisson renewal process is the one which has a linear renewal function.

4 THE RENEWAL EQUATION

Coming back to relation (3.9), we get, using the associative property of the convolution product:

$$\begin{aligned} H(t) &= F(t) + F^{(2)}(t) + F^{(3)}(t) + \cdots \\ &= F + F \bullet \left[F + F^{(2)} + \cdots\right](t) \\ &= F(t) + F \bullet H(t), \text{with } F \bullet H(t) = \int_0^t F(t-x)dH(x). \end{aligned} \tag{4.1}$$

This relation is called the integral equation of renewal theory, or simply the *renewal equation*. It can also be written as follows:

$$H(t) = F(t) + \int_0^t F(t-x)dH(x). \tag{4.2}$$

As:

$$F \bullet H(t) = H \bullet F(t), \tag{4.3}$$

we also have:

$$H(t) = F(t) + \int_0^t H(t-x)dF(x). \tag{4.4}$$

In the particular case where the density function f of F exists, the last integral equation becomes

$$H(t) = F(t) + \int_0^t H(t-x)f(x)dx. \tag{4.5}$$

In this case, we can apply the dominated convergence to series (3.9), showing the existence of the density function of H, say h, such that:

$$h(t) = \sum_{n=1}^{\infty} f^{[n]}(t), \tag{4.6}$$

with

$$f^{[1]}(t) = f(t), \tag{4.7}$$

$$f^{[2]}(t) = \int_0^t f(t-x)f(x)dx, \tag{4.8}$$

$$\vdots$$

$$f^{[n]}(t) = \int_0^t f^{[n-1]}(t-x)f(x)dx. \tag{4.9}$$

From relation (4.5) or (4.6) we obtain the integral equation for h:

$$h(t) = f(t) + f \otimes h(t), \tag{4.10}$$

with

$$f \otimes h(t) = \int_0^t f(t-x)h(x)dx. \tag{4.11}$$

As:

$$f \otimes h(t) = h \otimes f(t), \tag{4.12}$$

we also have:

$$h(t) = f(t) + h \otimes f(t). \tag{4.13}$$

In fact, the renewal equation (4.2) is one particular case of the type of integral equations:

$$X(t) = G(t) + X \bullet F(t), \tag{4.14}$$

where X is the unknown function, F and G being known measurable functions bounded on finite intervals and $\bullet$ the convolution product.

Such an integral equation is said to be of *renewal type*.

When $G = F$, we get the renewal equation. The study of these integral equations has a long history which includes contributions from Lotka (1940), Feller (1941), Smith (1954) and Çinlar (1969). Çinlar gave the two following propositions.

Proposition 4.1 (*Existence and unicity*)
The integral equation of renewal type (4.14) *has one and only one solution, given by*

$$X(t) = R \bullet G(t), \tag{4.15}$$

R being defined by relation (3.13).

It is also possible to study the asymptotic behaviour of solutions to renewal-type equations. The basic result is the so-called "key renewal theorem", proven by W.L. Smith (1954), and which is in fact mathematically equivalent to Blackwell's theorem (1948), given here as **Corollary 4.3**.
A proof of the key renewal theorem using Blackwell's theorem can be found in Çinlar (1975b).

Proposition 4.2 (*Asymptotic behaviour, Key renewal theorem)*
(i) *In the transient case, we have*:

$$\lim_{t\to\infty} X(t) = R(\infty)G(\infty) \tag{4.16}$$

provided the limit

$$G(\infty) = \lim_{t} G(t) \tag{4.17}$$

exists.
(ii) *In the case of recurrence, we have*:

$$\lim_{t\to\infty} X(t) = \frac{1}{m}\int_0^\infty G(x)dx, \tag{4.18}$$

provided that G is directly Riemann integrable on $[0,\infty)$, *that F is not arithmetic, and supposing*:

$$m = E(X_n) = \int_0^\infty (1 - F(x))dx\,. \tag{4.19}$$

Corollary 4.1 *In the case of a recurrent renewal process with finite variance* σ^2, *we have*:

$$\lim_{t\to\infty}\left(R(t) - \frac{t}{m}\right) = \frac{m^2 + \sigma^2}{2m^2}\,. \tag{4.20}$$

:
Remark 4.1
1) From result (4.20), we immediately get an analogous result for the renewal function *H*. Indeed, we know, from relation (3.14), that

$$R(t) = H(t) + U_0(t), \qquad t \geq 0. \tag{4.21}$$

Applying result (4.20), we get:

$$\lim_{t\to\infty}\left(H(t) - \frac{t}{m}\right) = \frac{m^2 + \sigma^2}{2m^2} - 1, \tag{4.22}$$

or

$$\lim_{t\to\infty}\left(H(t)-\frac{t}{m}\right)=\frac{\sigma^2-m^2}{2m^2}. \tag{4.23}$$

2) The two results (4.20) and (4.23) are often written under the following forms:

$$R(t)=\frac{t}{m}+\frac{m^2+\sigma^2}{2m^2}+\mathrm{O}(1), \tag{4.24}$$

$$H(t)=\frac{t}{m}+\frac{\sigma^2-m^2}{2m^2}+\mathrm{O}(1), \tag{4.25}$$

where O(1) represents a function of *t* approaching zero as *t* approaches infinity.

Corollary 4.2 *In the case of a recurrent renewal process with finite mean m, we have*:

$$\lim_{t\to\infty}\frac{R(t)}{t}=\frac{1}{m}. \tag{4.26}$$

Corollary 4.3 (*Blackwell's theorem*)
In the case of a recurrent process with finite mean m, we have, for every positive number τ:

$$\lim_{t\to\infty}\left(R(t)-R(t-\tau)\right)=\frac{\tau}{m}. \tag{4.27}$$

Remarks 4.2

1) *Probabilistic interpretation of renewal density.*

Let $k(t)dt$ represent the probability that there is a renewal in the time interval $(t,t+dt)$. It must satisfy the following relation, obtained by a simple probabilistic argument using the independence property of the successive "lifetimes":

$$k(t)dt=f(t)dt+\int_0^t f(x)k(t-x)dt. \tag{4.28}$$

From the unicity part of **Proposition 4.1**, we get, for all $t\geq 0$:

$$k(t)=h(t). \tag{4.29}$$

So, the probability defined above is given by $h(t)dt$ and more generally by $dH(t)$ with a precision error of $\mathrm{O}(dt)$. This interpretation often simplifies the search for relations useful in renewal theory.

2) *The variance of* $N(t)$

From Stein's lemma, we know that $N(t)$ has, for all *t*, moments of any order. Also, let $\alpha_2(t)$ be the centred moment of order 2 of $N(t)$:

$$\alpha_2(t)=E\left(\left(N(t)\right)^2\right). \tag{4.30}$$

From results (3.5), we can write successively:

$$\alpha_2(t) = \sum_{k=1}^{\infty} k^2 \left(U_0 - F \right) \bullet F^{(\nu)}(t) \tag{4.31}$$

$$= \sum_{k=1}^{\infty} k^2 F^{(k)}(t) - \sum_{k=1}^{\infty} k^2 F^{(k+1)}(t) \tag{4.32}$$

$$= \sum_{k=1}^{\infty} k^2 F^{(k)}(t) - \sum_{\nu=1}^{\infty} (\nu - 1)^2 F^{(\nu)}(t) \tag{4.33}$$

$$= \sum_{\nu=1}^{\infty} \left[\nu^2 - (\nu - 1)^2 \right] F^{(\nu)}(t) \tag{4.34}$$

$$= \sum_{\nu=1}^{\infty} (2\nu - 1) F^{(\nu)}(t) \tag{4.35}$$

$$= 2\sum_{\nu=1}^{\infty} (\nu - 1) F^{(\nu)}(t) + \sum_{\nu=1}^{\infty} F^{(\nu)}(t). \tag{4.36}$$

Now if we compute $H^{(2)}(t)$ by means of the relation:

$$H^{(2)}(t) = \left(\sum_{\nu=1}^{\infty} F^{(\nu)}(t) \right) \bullet \left(\sum_{\nu'=1}^{\infty} F^{(\nu')}(t) \right), \tag{4.37}$$

we easily find:

$$H^{(2)}(t) = \sum_{\nu=1}^{\infty} (\nu - 1) F^{(\nu)}(t). \tag{4.38}$$

Using (3.9) and substituting this last result in relation (4.36), we finally obtain:

$$\alpha^2(t) = H(t) + 2H^{(2)}(t). \tag{4.39}$$

This last result shows that

$$\mathrm{Var}\left(N(t) \right) = H(t) + 2H^{(2)}(t) - \left(H(t) \right)^2. \tag{4.40}$$

Remark 4.3

The renewal equation (4.4) can be solved, as it will be shown in the previous sections, directly in some very special cases or by means of the Laplace or Laplace-Stieltjes transforms in other cases.
In this way the analytical solution of the integral equation (4.4) can be obtained. But in the largest part of real life applications the equation that is obtained by the model can't be solved by analytical methods. In these cases it is necessary to use a numerical approach to get the solution to the general renewal equation (4.4) in a bounded horizon time (see Janssen and Manca(2006), Chapter 2, section 10)

5 THE USE OF LAPLACE TRANSFORM

5.1 The Laplace Transform

To show the power of the Laplace transform in solving the renewal equation, let us suppose that d.f. F characterizing the recurrent renewal process considered has f as density.

Let us use the following general notation: for any function α on $[0,\infty)$, $\tilde{\alpha}$ will represent its Laplace transform, provided it exists:

$$\tilde{\alpha}(s)=\int_0^\infty e^{-sx}\alpha(x)dx,\left(=\tilde{£}\left(\alpha(x)\right)\right). \tag{5.1}$$

With this convention and using the well-known property of the Laplace transform

$$\tilde{£}\left((\alpha\otimes\beta)(x)\right)=\tilde{\alpha}(s)\cdot\tilde{\beta}(s), \tag{5.2}$$

we get from the renewal equation:

$$\tilde{h}(s)=\tilde{f}(s)+\tilde{h}(s)\cdot\tilde{f}(s). \tag{5.3}$$

The Laplace transform of the unique solution is thus given by:

$$\tilde{h}(s)=\frac{\tilde{f}(s)}{1-\tilde{f}(s)}. \tag{5.4}$$

Using the inverse Laplace transform, we can then have the value of $h(t)$.

Remark 5.1 From the algebraic equation (5.3), we can deduce the expression of the density f as a function of the renewal density h. In Laplace transforms, we get:

$$\tilde{f}(s)=\frac{\tilde{h}(s)}{1-\tilde{h}(s)}. \tag{5.5}$$

The inverse Laplace transform gives us a function of h.

This leads to the important result that *every renewal process is characterized by its renewal density, if it exists, or by its renewal function*. Thus there is a one-to-one correspondence between the d.f. F of a renewal function and its renewal function H.

5.2 The Laplace-Stieltjes (L-S) Transform

The L-S transform is a bit more general than the Laplace transform. Indeed, for any function α on $[0,\infty)$, $\bar{\alpha}$ will represent its L-S transform, provided it exists, under the form:

$$\overline{\alpha}(s)=\int_0^{\infty}e^{-sx}d\alpha(x)\qquad\left(=\overline{\text{£}}(\alpha(x))\right)\ . \tag{5.6}$$

For a function α such that

$$\lim_{x\to\infty}e^{-sx}\alpha(x)=0\ , \tag{5.7}$$

an integration by parts gives the relation between $\tilde{\alpha}$ and $\overline{\alpha}$:

$$\overline{\alpha}(s)=-\alpha(0)+s\tilde{\alpha}(s). \tag{5.8}$$

As $\overline{\text{£}}$ satisfies the following property:

$$\overline{\text{£}}((\alpha\bullet\beta)(x))=\overline{\text{£}}(\alpha(x))\cdot\overline{\text{£}}(\beta(x))\ , \tag{5.9}$$

we get from the renewal equation (4.4):

$$\overline{H}(s)=\overline{F}(s)+\overline{H}(s)\cdot\overline{F}(s). \tag{5.10}$$

Or:

$$\overline{H}(s)=\frac{\overline{F}(s)}{1-\overline{F}(s)}\ , \tag{5.11}$$

which is equivalent to the expression (5.4) if we do not assume the existence of a density for F. Of course, if such a density exists, we have from (5.6):

$$\overline{F}(s)=\tilde{f}(s), \tag{5.12}$$

$$\overline{H}(s)=\tilde{h}(s), \tag{5.13}$$

and consequently, the relations (5.11) and (5.4) are identical.

6 APPLICATION OF WALD'S IDENTITY

We already know that

$$E(S_n)=nm, \tag{6.1}$$

as

$$S_n=X_0+\cdots+X_n \tag{6.2}$$

and

$$E(N(t))=R(t). \tag{6.3}$$

Now let us consider the time of the first "replacement" after time t; it is given by $S_{N'(t)}$. Wald's lemma (see Chapter 1, relation (6.52)) computes the mean of this random variable.

Proposition 6.1 (*Wald's lemma*) $E\left(S_{N'(t)}\right)=mR(t).$ (6.4)

Remark 6.1 As, by (3.10):

$$N'(t)=N(t)+1\ , \tag{6.5}$$

we can use relation (3.14) to write result (6.4) as:

$$E\left(S_{N(t)+1}\right)=m\left[H(t)+1\right]\ . \tag{6.6}$$

7 ASYMPTOTIC BEHAVIOUR OF THE $N(t)$-PROCESS

This section will yield two important results concerning the counting process $(N(t), t \geq 0)$ associated with a recurrent renewal process characterized by the d.f. F.

Proposition 7.1 (*Strong law of large numbers*) *If* $m < \infty$, *then, almost surely*:

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{m}. \tag{7.1}$$

Proposition 7.2 (*Central limit theorem*) *If* $\sigma^2 < \infty$, *then for all* $y \in \mathbb{R}$:

$$\lim_{t \to \infty} P\left(\frac{N(t) - t/m}{\sqrt{\sigma^2 \, t/m^3}} \leq y \right) = \Phi(y) \ , \tag{7.2}$$

where Φ *represents the standard normal d.f.*.

Remark 7.1 From **Proposition 7.2**, we have the following approximation for large t:

$$\operatorname{var}(N(t)) \sim \frac{\sigma^2}{m^3} t, \tag{7.3}$$

which simplifies the exact result (4.40).

8 DELAYED AND STATIONARY RENEWAL PROCESSES

The notion of *stationary renewal process* is a particular case of a *delayed renewal process.* A delayed renewal process is a renewal process with the difference that the first r.v. X_1, though still independent of the others, does not have the same distribution.

More precisely, let $(X_n, \ n \geq 1)$ be a sequence of non-negative independent variables, G being the d.f. of X_1 and F the d.f. of all other r.v.

The corresponding sequence $(T_n, \ n \geq 0)$, where

$$T_0 = 0 \quad \text{a.s.}, \tag{8.1}$$

$$T_n = X_1 + \cdots + X_n, \tag{8.2}$$

is called a *delayed renewal sequence or delayed renewal process.*

Clearly the basic definition of the "classical" renewal processes can be extended to the case of a delayed renewal process. For example if $H_d(t)$ represents the renewal function for a delayed renewal process, and if we pose the condition $X_1 = x,$ then we have:

$$H_d(t \mid X_1 = x) = \begin{cases} 0 & , \quad x > t, \\ 1 + H(t-x) & , \quad x \le t, \end{cases} \tag{8.3}$$

where H represents the renewal function associated with the d.f. F. Since, by definition:

$$H_d(t \mid X_1) = E(N(t) | X_1), \tag{8.4}$$

we get:

$$E(N(t)) = E(H_d(t, X_1)) \tag{8.5}$$

$$= \int_0^t [1 + H(t-x)] dG(x). \tag{8.6}$$

Or:

$$H_d = G(t) + H \bullet G(t). \tag{8.7}$$

So, knowing renewal function $H,$ we are just a convolution product away from knowing H_d.

This notion of delayed renewal process is introduced because it has been remarked that if one begins to observe a renewal process which has been running for a long time, the first variable X_1 observed has the d.f.

$$G(x) = \frac{1}{m}\int_0^x [1 - F(u)] du, \qquad x \ge 0. \tag{8.8}$$

If we consider a delayed renewal process with the d.f. defined by relation (8.8) for **G** the d.f. of X_1, it called a *stationary renewal process.*

9 MARKOV CHAINS

This section presents briefly some fundamental results concerning the theory of Markov chains with a finite number of states. These results will be used in the following chapter. We will use the usual terminology introduced by Chung (1960) and Parzen (1962).

9.1 Definitions

Let us consider an economic or physical system S with m possible states, represented by the set I:

$$I = \{1, \ 2, \ldots, \ m\}. \tag{9.1}$$

Let the system S evolve randomly in discrete time $(t = 0, \ 1, \ 2, \ldots, \ n, \ldots)$, and let J_n be the r.v. representing the state of the system S at time n.

Definition 9.1 *The random sequence* $(J_n, \quad n \in \mathrm{N})$ *is a Markov chain iff for all* $j_0,\ j_1,\ldots,j_n \in I$:

$$P(J_n = j_n \mid J_0 = j_0,\ J_1 = j_1,\ldots,\ J_{n-1} = j_{n-1}) = P(J_n = j_n \mid J_{n-1} = j_{n-1}) \quad (9.2)$$

(*provided this probability has meaning*).

Definition 9.2 *A Markov chain* $(J_n, \ n \geq 0)$ *is homogeneous iff the probabilities* (1.2) *do not depend on* n *and is non-homogeneous in the other cases.*

For the moment, we will only consider the homogeneous case for which we write:

$$P(J_n = j \mid J_{n-1} = i) = p_{ij}, \quad (9.3)$$

and we introduce the matrix **P** defined as:

$$\mathbf{P} = [\,p_{ij}\,]. \quad (9.4)$$

The elements of the matrix **P** have the following properties:

(i) $p_{ij} \geq 0$, for all $i, j \in I$, (9.5)

(ii) $\sum_{j \in I} p_{ij} = 1$, for all $i \in I$. (9.6)

A matrix **P** satisfying these two conditions is called a *Markov matrix* or a *transition matrix*.

To every transition matrix, we can associate a *transition graph* where vertices represent states. There exists an *arc* between vertices *i* and *j* iff $p_{ij} > 0$.

To fully define the evolution of a Markov chain, it is also necessary to fix an *initial distribution* for state J_0, i.e. a vector

$$\mathbf{p} = (p_1,\ldots,p_m), \quad (9.7)$$

such that:

$$p_i \geq 0, \qquad i \in I, \quad (9.8)$$

$$\sum_{i \in I} p_i = 1. \quad (9.9)$$

For all i, p_i represents the *initial probability* of starting from i :

$$p_i = P(J_0 = i). \quad (9.10)$$

For the rest of this chapter we will consider homogeneous Markov chains as being characterized by the couple $(\mathbf{p}, \mathbf{P})$.

If $J_n = i$ a.s., that is if the system starts with probability equal to 1 from state i, then the components of vector **p** will be:

$$p_j = \delta_{ij}. \quad (9.11)$$

We now introduce the *transition probabilities of order* $p_{ij}^{(n)}$, defined as:

$$p_{ij}^{(n)} = P(J_{\nu+n} = j \mid J_\nu = i). \tag{9.12}$$

From the Markov property (9.2), it is clear that conditioning with respect to $J_{\nu+1}$, we get

$$p_{ij}^{(2)} = \sum_k p_{ik} p_{kj}. \tag{9.13}$$

Using the following matrix notation:

$$\mathbf{P}^{(2)} = \left[p_{ij}^{(2)} \right], \tag{9.14}$$

we find that relation (9.13) is equivalent to

$$\mathbf{P}^{(2)} = \mathbf{P}^2. \tag{9.15}$$

Using induction, it is easy to prove that if

$$\mathbf{P}^{(n)} = \left[p_{ij}^{(n)} \right], \tag{9.16}$$

then we obtain for all $n \geq 1$:

$$\mathbf{P}^{(n)} = \mathbf{P}^n. \tag{9.17}$$

Note that (9.17) implies that the transition probability matrix in n steps is equal to the nth power of the matrix **P**.
For the marginal distributions related to J_n, we define for $i \in I$ and $n \geq 0$:

$$p_i(n) = P(J_n = i). \tag{9.18}$$

These probabilities may be computed as follows:

$$p_i(n) = \sum_j p_j p_{ji}^{(n)}, \qquad i \in I. \tag{9.19}$$

If we write:

$$p_{ji}^{(0)} = \delta_{ji} \text{ or } \mathbf{P}^{(0)} = \mathbf{I}, \tag{9.20}$$

then relation (9.19) is true for all $n \geq 0$.
If:

$$\mathbf{p}(n) = (p_1(n), \ldots, p_m(n)), \tag{9.21}$$

then relation (9.19) can be expressed, using matrix notation, as:

$$\mathbf{p}(n) = \mathbf{p}\mathbf{P}^n. \tag{9.22}$$

Definition 9.3 *A Markov matrix* **P** *is regular if there exists a positive integer* k*, such that all the elements of the matrix* $\mathbf{P}^{(k)}$ *are strictly positive.*

From relation (9.17), **P** is regular iff there exists an integer $k > 0$ such that all the elements of the kth power of **P** are strictly positive.

Example 9.1 (i) If:

$$\mathbf{P} = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix} \tag{9.23}$$

we have:

$$\mathbf{P}^2 = \begin{bmatrix} .75 & .25 \\ .5 & .5 \end{bmatrix} \tag{9.24}$$

so that **P** is regular.
The transition graph associated to **P** is given in **Figure 9.1**.

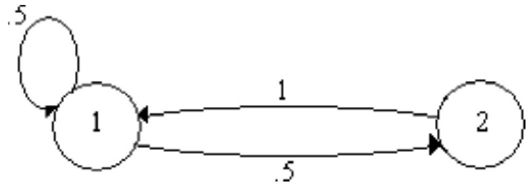

Figure 9.1

(ii) If:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ .75 & .25 \end{bmatrix}, \tag{9.25}$$

P is not regular, because for any integer k,

$$p_{12}^{(k)} = 0. \tag{9.26}$$

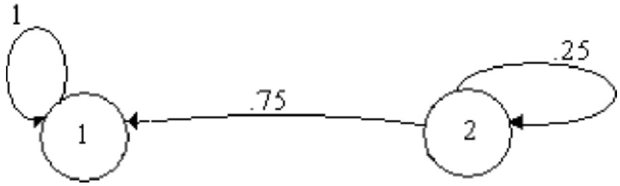

Figure 9.2

The transition graph in this case is depicted in **Figure 9.2**.
The same is true for the matrix:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{9.27}$$

(iii) Any matrix **P** whose elements are all strictly positive is regular.
For example:

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \qquad \begin{bmatrix} .7 & .2 & .1 \\ .6 & .2 & .2 \\ .4 & .1 & .5 \end{bmatrix}. \tag{9.28}$$

9.2. Markov Chain State Classification

Let $i \in I$, and let $d(i)$ be the greatest common divisor of the set of integers n, such that

$$p_{ii}^{(n)} > 0. \qquad (9.29)$$

Definition 9.4 *If $d(i) > 1$, the state i is said to be periodic with period $d(i)$. If $d(i) = 1$, then state i is aperiodic.*

Clearly, if $p_{ii} > 0$, then i is aperiodic. However, the converse is not necessarily true.

Remark 9.1 If **P** is regular, then all the states are aperiodic.

Definition 9.5 *A Markov chain whose states are all aperiodic is called an aperiodic Markov chain.*

From now on, we will have only Markov chains of this type.

Definition 9.6 *A state i is said to lead to state j (written $i \rhd j$) iff there exists a positive integer n such that*

$$p_{ij}^{n} > 0. \qquad (9.30)$$

i C j means that i does not lead to j.

Definition 9.7 *States i and j are said to communicate iff $i \rhd j$ and $j \rhd i$, or if $j = i$. We write $i \lhd\rhd j$.*

Definition 9.8 *A state i is said to be essential iff it communicates with every state it leads to; otherwise it is called inessential.*

The relation $\lhd\rhd$ defines an equivalence relation over the state space I resulting in a partition of I. The equivalence class containing state i is represented by $C(i)$.

Definition 9.9 *A Markov chain is said to be irreducible iff there exists only one equivalence class.*

Clearly, if **P** is regular, the Markov chain is both irreducible and aperiodic. Such a Markov chain is said to be *ergodic*.

It is easy to show that if the state i is essential (inessential) then all the elements of the class $C(i)$ are essential (inessential) (see Chung (1960)).
We can thus speak of essential and inessential classes.

Definition 9.10 *A subset E of the state space I is said to be closed iff:*

$$\sum_{j\in E} p_{ij} = 1, \text{ for all } i \in E. \tag{9.31}$$

It can be shown that every essential class is minimally closed. See Chung (1960).

Definition 9.11 *For given states i and j, with $J_0 = i$, we can define the r.v. τ_{ij} called the first passage time to state j as follows:*

$$\tau_{ij} = \begin{cases} n & \text{if } J_\nu \neq j, \quad 0 < \nu < n, \quad J_n = j, \\ \infty & \text{if } J_\nu \neq j, \quad \text{for all } \nu > 0. \end{cases} \tag{9.32}$$

τ_{ij} is said to be the *hitting time* of the singleton $\{j\}$, starting from state i at time 0.
Supposing:

$$f_{ij}^{(n)} = P(\tau_{ij} = n \mid J_0 = i), \qquad n \in \mathbb{N}_0 \tag{9.33}$$

and

$$f_{ij} = P(\tau_{ij} < \infty \mid J_0 = i), \tag{9.34}$$

one can see that for $n > 0$:

$$f_{ij}^{(n)} = P(J_n = j, \quad J_\nu \neq j, \quad 0 < \nu < n \mid J_0 = i), \tag{9.35}$$

$$= \sum_{S'_{n,i,j}} \prod_{k=0}^{n-1} p_{\alpha_k \alpha_{k+1}}, \tag{9.36}$$

where the summation set $S'_{n,i,j}$ is defined as:

$$S'_{n,i,j} = \{(\alpha_0, \alpha_1, \ldots, \alpha_n): \alpha_0 = i, \alpha_n = j, \alpha_k \in I, \alpha_k \neq j, \quad k = 1, \ldots, n-1\}. \tag{9.37}$$

We also have:

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}, \tag{9.38}$$

$$1 - f_{ij} = P(\tau_{ij} = \infty \mid J_0 = i). \tag{9.39}$$

The elements $f_{ij}^{(n)}$ can readily be computed by induction, using the following relations:

$$p_{ij} = f_{ij}^{(1)}, \tag{9.40}$$

$$p_{ij}^{(n)} = \sum_{\nu=1}^{n-1} f_{ij}^{(\nu)} p_{jj}^{(n-\nu)} + f_{ij}^{(n)}, \qquad n \geq 2. \tag{9.41}$$

Let:

$$m_{ij} = E\left(\tau_{ij} \mid J_0 = i\right), \tag{9.42}$$

with the possibility of an infinite mean. The value of m_{ij} is given by:

$$m_{ij} = \sum_{n=1}^{\infty} n f_{ij}^{(n)} - \infty\left(1 - f_{ij}\right).^{(*)} \tag{9.43}$$

If $i = j$, then m_{ij} is called the *first passage time mean* or the *mean recurrence* time of state i.

For every j, we define the sequence of successive return times to state j $\left(r_n^{(j)},\quad n \geq a\right)$ as follows:

$$r_0^{(j)} = 0, \tag{9.44}$$

$$r_n^{(j)} = \sup_k\left\{k \in \mathrm{N}_{0,}\quad k > r_{n-1}^{(j)},\ J_\nu \neq j,\ r_{n-1}^{(j)} < \nu < k\right\},\quad n > 0. \tag{9.45}$$

Using the Markov property and supposing $J_0 = j$, the sequence of return times to state j is a renewal sequence with the r.v.

$$r_n^{(j)} - r_{n-1}^{(j)},\qquad n \geq 1 \tag{9.46}$$

distributed according to τ_{jj}.

If $J_0 = i,\quad i \neq j,$ then $\left(r_n^{(j)},\quad n \geq 0\right)$ is a general renewal sequence. In this case:

$$r_1^{(j)} = \tau_{ij}, \tag{9.47}$$

and

$$r_n^{(j)} - r_{n-1}^{(j)} \sim \tau_{jj},\qquad n > 1. \tag{9.48}$$

This shows that a Markov chain contains many embedded renewal processes. These processes are used to define the next classification of states.

Definition 9.12 *A state i is said to be transient* (*recurrent*) *if the renewal process associated with its successive return times to i is transient* (*recurrent*).

A direct consequence of this definition is that:

$$i\ \text{ transient } \Leftrightarrow\ f_{ii} < 1, \tag{9.49}$$

$$i\ \text{ recurrent } \Leftrightarrow\ f_{ii} = 1. \tag{9.50}$$

A recurrent state i is said to be *null* (*positive*) if $m_{ii} = \infty\quad\left(m_{ii} < \infty\right)$. It can be shown that if $m_{ii} < \infty$, then we can only have positive recurrent states.

This classification leads to the decomposition theorem (see Chung (1960)).

(*) Using the following conventions:
$\infty + a = \infty,\ \ a \in \mathbb{R},\ \ \infty \cdot a = \infty,\ \ (a > 0)$, and in this particular case, $\infty \cdot 0 = 0$.

Proposition 9.1 (*Decomposition theorem*) *The state space I of any Markov chain can be decomposed into* r $(r \geq 1)$ *subsets* $C_1,\ldots,C_r$ *forming a partition, such that each subset* C_i *is one and only one of the following types*:

(i) *an essential recurrent positive closed set,*

(ii) *an inessential transient non-closed set.*

Remark 9.2

(1) If an inessential class reduces to a singleton $\{i\}$, there are two possibilities:

a) There exists a positive integer N such that:

$$0 < p_{ii}^{N} < 1. \tag{9.51}$$

b) The N in a) does not exist. In this case, the state i is said to be a *non-return state*.

(2) If the singleton $\{i\}$ forms an essential class, then

$$p_{ii} = 1 \tag{9.52}$$

and the state i is said to be an *absorbing state*.

(3) If $m = \infty$, there may be two other types of class in the decomposition theorems:

a) *essential transient closed,*

b) *essential recurrent non-closed classes.*

The literature on Markov chains gives the following necessary and sufficient conditions for recurrence and transience.

Proposition 9.2

(i) *State i is transient iff*

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty. \tag{9.53}$$

In this case, for all $k \in I$:

$$\sum_{n=1}^{\infty} p_{ki}^{(n)} < \infty, \tag{9.54}$$

and in particular:

$$\lim_{n\to\infty} p_{ki}^{(n)} = 0, \quad \forall k \in I. \tag{9.55}$$

(ii) *State i is recurrent iff*

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty. \tag{9.56}$$

In this case:

$$k \lhd\rhd i \Rightarrow \sum_{n=1}^{\infty} p_{ki}^{(n)} = \infty, \tag{9.57}$$

and

$$k \, \mathsf{C} \, i \Rightarrow \sum_{n=1}^{\infty} p_{ki}^{(n)} = 0. \tag{9.58}$$

9.3 Occupation Times

For any state j, and for $n \in \mathbb{N}_0$, we define the r.v. $N_j(n)$ as the number of times the state j is occupied in the first n transitions:

$$N_j(n) = \#\{k \in \{1,\ldots,n\} : J_k = j\}. \tag{9.59}$$

By definition, the r.v. $N_j(n)$ is called the *occupation time of state j in the first n transitions.*
The r.v.

$$N_j(\infty) = \lim_{n\to\infty} N_j(n) \tag{9.60}$$

is called the *total occupation time of state j.*
For any state j and $n \in \mathbb{N}_0$ let us define:

$$Z_j(n) = \begin{cases} 1 & \text{if } J_n = j, \\ 0 & \text{if } J_n \neq j. \end{cases} \tag{9.61}$$

We may write:

$$N_j(n) = \sum_{\nu=1}^{n} Z_j(\nu). \tag{9.62}$$

We have from relation (9.34):

$$\mathrm{P}\left(N_j(\infty) > 0 \mid J_0 = i\right) = f_{ij}. \tag{9.63}$$

Let g_{ij} be the conditional probability of an infinite number of visits to the state j, starting with $J_0 = i$; that is:

$$g_{ij} = P\left(N_j(\infty) = \infty \mid J_0 = i\right). \tag{9.64}$$

It can be shown that:

$$g_{ii} = \lim_{n\to\infty} f_{ii}^{(n)}, \tag{9.65}$$

$$g_{ij} = f_{ij} \cdot g_{jj}, \tag{9.66}$$

$$g_{ii} = 1 \Leftrightarrow f_{ii} = 1 \Leftrightarrow i \text{ is recurrent,} \tag{9.67}$$

$$g_{ii} = 0 \Leftrightarrow f_{ii} < 1 \Leftrightarrow i \text{ is transient.} \tag{9.68}$$

Results (9.67) and (9.68) can be interpreted as showing that the system S will visit a recurrent state an infinite number of times, and that it will visit a transient state a finite number of times.

9.4 Computation Of Absorption Probabilities

Proposition 9.3

(i) *If i is recurrent and if* $j \in C(i)$, *then* $f_{ij} = 1$.

(ii) *If i is recurrent and if* $j \notin C(i)$, *then* $f_{ij} = 0$.

Proposition 9.4 *Let T be the set of all transient states of I, and let C be a recurrent class.*
For all $j,\ k \in C$,

$$f_{ij} = f_{ik}. \tag{9.69}$$

Labeling this common value as f_{iC}, *the probabilities* $\left(f_{i,C},\ \ i \in T \right)$ *satisfy the linear system*:

$$f_{i,C} = \sum_{k \in T} p_{ik} f_{k,C} + \sum_{k \in C} p_{ik}, \qquad i \in T. \tag{9.70}$$

Remark 9.3 Parzen (1962) proved that under the assumption of **Proposition 9.4**, the linear system (9.70) has a unique solution. This shows, in particular, that if there is only one irreducible class C, then for all $i \in T$:

$$f_{i,C} = 1. \tag{9.71}$$

Definition 9.13 *The probability* $f_{i,C}$ *introduced in* **Proposition 9.4** *is called absorption probability in class C, starting from state i.*

If class C is recurrent:

$$f_{i,C} = \begin{cases} 1 & \text{if } i \in C, \\ 0 & \text{if } i \text{ is recurrent, } \ i \notin C. \end{cases} \tag{9.72}$$

9.5 Asymptotic Behaviour

Consider an irreducible aperiodic Markov chain which is positive recurrent.
Suppose that the following limit exists:

$$\lim_{n \to \infty} p_j(n) = \pi_j, \quad j \in I \tag{9.73}$$

starting with $J_0 = i$.
The relation

$$p_j(n+1) = \sum_{k \in I} p_k(n) p_{kj} \tag{9.74}$$

becomes:

$$p_{ij}^{(n+1)} = \sum_{k \in I} p_{ik}^{(n)} p_{kj}, \tag{9.75}$$

because

$$p_j(n) = p_{ij}^{(n)}. \tag{9.76}$$

Since the state space I is finite, we obtain from (9.73) and (9.75):

$$\pi_j = \sum_{k \in I} \pi_k p_{kj} \,, \tag{9.77}$$

and from (9.76):

$$\sum_{i \in I} \pi_i = 1 \,. \tag{9.78}$$

The result:

$$\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j \tag{9.79}$$

is called an *ergodic result*, since the value of the limit in (9.79) is independent of the initial state i.

From result (9.79) and (9.19), we see that for any initial distribution $\mathbf{p}$:

$$\lim_{n \to \infty} p_i(n) = \lim_{n \to \infty} \sum_j p_j p_{ji}^{(n)}, \tag{9.80}$$

$$= \sum_j p_j \pi_i \,, \tag{9.81}$$

so that:

$$\lim_{n \to \infty} p_i(n) = \pi_i \,. \tag{9.82}$$

This shows that the asymptotic behaviour of a Markov chain is given by the existence (or non-existence) of the limit of the matrix $\mathbf{P}^n$.

A standard result concerning the asymptotic behaviour of $\mathbf{P}^n$ is given in the next proposition. The proof can be found in Chung (1960), Parzen (1962) or Feller (1957).

Proposition 9.5 *For any aperiodic Markov chain of transition matrix* $\mathbf{P}$ *and having a finite number of states, we have*:

a) *if state* j *is recurrent (necessarily positive), then*

(i) $i \in C(j) \Rightarrow \lim_{n \to \infty} p_{ij}^{(n)} = \dfrac{1}{m_{jj}}$, (9.83)

(ii) *i recurrent and* $\notin C(j) \Rightarrow \lim_{n \to \infty} p_{ij}^{(n)} = 0$, (9.84)

(iii) *i transient* $\lim_{n \to \infty} p_{ij}^{(n)} = \dfrac{f_{i,C(j)}}{m_{jj}}$. (9.85)

b) *If j is transient, then for all* $i \in I$:

$$\lim_{n \to \infty} p_{ij}^{(n)} = 0. \tag{9.86}$$

Remark 9.4

Result (ii) of part a) is trivial since in this case:

$p_{ij}^{(n)} = 0$ for all positive *n*.

From **Proposition 9.5**, the following corollaries can be deduced.

Corollary 9.1 (*Irreducible case*) *If the Markov chain of transition matrix* **P** *is irreducible, then for all* $i, j \in I$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j, \tag{9.87}$$

with

$$\pi_j = \frac{1}{m_{jj}}. \tag{9.88}$$

It follows that for all j:

$$\pi_j > 0. \tag{9.89}$$

If we use **Remark 9.3** in the particular case where we have only one recurrent class and where the states are transient (the so-called *uni-reducible* case), then we have the following corollary:

Corollary 9.2 (*Uni-reducible case*) *If the Markov chain of transition matrix* **P** *has one essential class* C (*necessarily recurrent positive*) *and* T *as transient set, then we have*:

(i) for all $i,\ \ j \in C$: $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j,$ (9.90)

with $\{\pi_j, j \in C\}$ *being the unique solution of the system*:

$$\pi_j = \sum_{i\in C} \pi_i p_{ij}, \tag{9.91}$$

$$\sum_{j\in C} \pi_j = 1. \tag{9.92}$$

(ii) For all $j \in T$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = 0 \text{ for all } i \in I. \tag{9.93}$$

(iii) *For all* $j \in C$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j \text{ for all } i \in T. \tag{9.94}$$

Remark 9.5 Relations (9.91) and (9.92) are true because the set C of recurrent states can be seen as a Markov sub-chain of the initial chain.

If the ℓ transient states belong to the set $\{1,\ldots,\ell\}$, using a permutation of the set I, if necessary, then the matrix **P** takes the following form:

$$\begin{array}{c} 1 \\ \vdots \\ \ell \\ \ell+1 \\ \vdots \\ m \end{array} \overset{\begin{array}{cccccc} 1 & \cdots & \ell & \ell+1 & \cdots & m \end{array}}{\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{O} & \mathbf{P}_{22} \end{bmatrix}} = \mathbf{P}. \tag{9.95}$$

This proves that the sub-matrix $\mathbf{P}_{22}$ is itself a Markov transition matrix.

Let us now consider a Markov chain of matrix **P.** The general case is given by a partition of I:

$$I = T \cup C_1 \cup \cdots \cup C_r, \tag{9.96}$$

where T is the set of transient states and $C_1,\ldots,C_r$ the r positive recurrent classes.

By reorganizing the order of the elements of I, we can always suppose that

$$T = \{1,\ldots,\ell\}, \tag{9.97}$$

$$C_1 = \{\ell+1,\ldots,\ell+\nu_1\}, \tag{9.98}$$

$$C_2 = \{\ell+\nu_1+1,\ldots,\ell+\nu_1+\nu_2\}, \tag{9.99}$$

$\vdots$

$$C_r = \left\{\ell + \sum_{j=1}^{r-1}\nu_j + 1,\ldots,\ m\right\}, \tag{9.100}$$

where ν_j is the number of elements in C_j, $(j = 1,\ldots,r)$ and

$$\ell + \sum_{j=1}^{r}\nu_j = m. \tag{9.101}$$

This results from the following block partition of matrix **P**:

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{\ell\times\ell} & \mathbf{P}_{\ell\times\nu_1} & \mathbf{P}_{\ell\times\nu_2} & \cdots & \mathbf{P}_{\ell\times\nu_r} \\ \mathbf{0} & \mathbf{P}_{\nu_1\times\nu_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_{\nu_2\times\nu_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{P}_{\nu_r\times\nu_r} \end{bmatrix} \tag{9.102}$$

where, for $j = 1,\ldots,\ r$:

$\mathbf{P}_{\ell\times\ell}$ is the transition sub-matrix for T,

$\mathbf{P}_{\ell\times\nu_j}$ is the transition sub-matrix from T to C_j,

$\mathbf{P}_{\nu_j\times\nu_j}$ is the transition sub-matrix for the class C_j.

From **Proposition 9.1**, we have the following corollary:

Corollary 9.3 *For a general Markov chain of matrix* $\mathbf{P}$, *given by* (9.102), *we have*:

(i) *For all* $i \in I$ *and all* $j \in T$: $\lim_{n\to\infty} p_{ij}^{(n)} = 0.$ (9.103)

(ii) *For all* $j \in C_\nu \quad (\nu = 1,\ldots,\ r)$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = \begin{cases} \pi_j & \text{if } i \in C_\nu, \\ 0 & \text{if } i \in C_{\nu'} \quad \nu' \neq \nu, \\ f_{i,C_\nu} \pi_j^\nu & \text{if } i \in T. \end{cases} \tag{9.104}$$

Moreover, for all $\nu = 1,\ldots,\ r$:

$$\sum_{j\in C_\nu} \pi_j^\nu = 1. \tag{9.105}$$

This last result allows us to calculate the limit values quite simply.
For $\left(\pi_j^\nu,\ j \in C_\nu\right),\ \nu = 1,\ldots,\ r$, it suffices to solve the linear systems for each fixed ν :

$$\begin{cases} \pi_j^\nu = \sum_{k\in C_\nu} \pi_k^\nu p_{kj}, \quad j \in C_\nu, \\ \sum_{i\in C_\nu} \pi_i^\nu = 1. \end{cases} \tag{9.106}$$

Indeed, since each C_ν is itself a space set of an irreducible Markov chain of matrix $\mathbf{P}_{\nu\times\nu}$, the above relations are none other than (9.77) and (9.78).
For the absorption probabilities $\left(f_{i,C_\nu},\ i \in T\right),\ \nu = 1,\ldots,\ r$, it suffices to solve the following linear system for each fixed ν. Using **Proposition 9.4**, we have:

$$f_{i,C_\nu} = \sum_{k\in T} p_{ik} f_{i,C_\nu} + \sum_{k\in C_\nu} p_{ik}, \qquad i \in T. \tag{9.107}$$

An algorithm, given in De Dominicis, Manca (1984b) very useful for the classification of the states of a Markov chain, is fully developed in Janssen and Manca (2006), section 8.

9.6 Examples

Markov chains appear in many practical problems in such fields as operations research, business, social sciences, etc.
To give an idea of this potential, we will present some simple examples followed by a fully developed case study in the domain of social insurance.

(i) *A transportation problem*. (Anton & Kolman (1978)).
Let us consider a taxicab company of a city V, subdivided into three sectors $V_1,\ V_2$ and V_3.

A taxicab picks up a passenger in any sector and drops her or him off in any sector.
We can view a taxicab as a physical system S which can be in one of three states: the sectors V_1, V_2 or V_3.
The observation of taxicabs leads to the construction of a Markov chain with three states.
This Markov chain might have the following matrix **P**, for example:

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.6 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}. \qquad (9.108)$$

This matrix is regular, hence irreducible and aperiodic since all its elements are strictly positive.

(ii) *A management problem in an insurance company*
A car insurance company classifies its customers in three groups:
G_0: Those having no accidents during the year,

G_1 : Those having one accident during the year,

G_2: Those having more than one accident during the year.
The statistics department of the company observes that the annual transition between the three groups can be represented by a Markov chain with state space $\{G_0, G_1, G_2\}$ and transition matrix **P**:

$$\mathbf{P} = \begin{bmatrix} .85 & .10 & .05 \\ 0 & .80 & .20 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (9.109)$$

We suppose that the company produces 50,000 new contracts per year and wants to know the distribution of these contracts for the next four years.
After one year, one has, on average:
in group G_0 : $50,000 \times .85 = 42,500$,

in group G_1 : $50,000 \times .10 = 5,000$,

in group G_2 : $50,000 \times .05 = 2,500$.
These results are simply the elements of the first row of **P,** multiplied by 50,000.
After two years, multiplying the elements of the first row of $\mathbf{P}^{(2)}$ by 50,000, we get
in group G_0 : $36,125$,

in group G_1 : $8,250$,

in group G_2 : $5,625$.
A similar computation gives:

	After three years	After four years
G_0	30,706	26,100
G_1	10,213	11,241
G_3	9,081	12,659

To find the type of the Markov chain with transition matrix (9.109), the simple graph of possible transitions given in **Figure 9.3** shows that the class $\{1,\ 2\}$ is transient and class $\{3\}$ is absorbing. Thus, using **Corollary 9.2** we obtain the limit matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.110}$$

The limit matrix can be interpreted as showing that regardless of the initial composition of the group; the customers will finish by having at least two accidents.

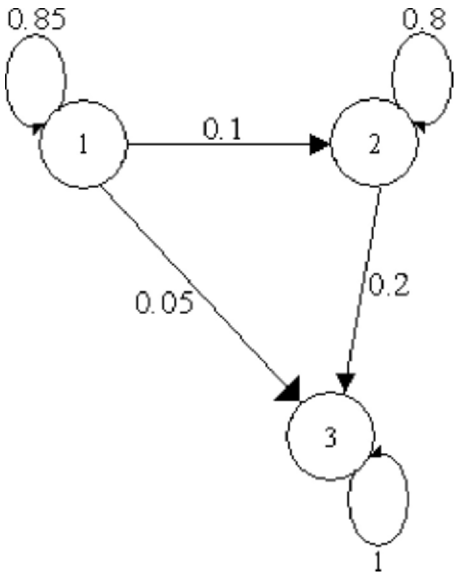

Figure 9.3

Remark 9.6 If one wants to know the situation after one or two changes, one can use relation (1.19) with $n = 1,2,3$ and with $\mathbf{p}$ given by:

$$\mathbf{p} = (.26,.60,.14)\,. \tag{9.111}$$

One obtains the following results:

$p_1^{(1)} = .257 \quad p_2^{(1)} = .597 \quad p_3^{(1)} = .146$
$p_1^{(2)} = .255 \quad p_2^{(2)} = .594 \quad p_3^{(2)} = .151$
$p_1^{(3)} = .254 \quad p_2^{(3)} = .590 \quad p_3^{(3)} = .156.$

These results show that the convergence of $\mathbf{p}^{(n)}$ to π is relatively fast.

9.7 A Case Study In Social Insurance (Janssen (1966))

To compute insurance or pension premiums for professional diseases such as silicosis, we need to compute the average (mean) degree of disability at pre-assigned time periods. Let us suppose we retain m degrees of disability:
$S_1, \ldots, S_m$, the last being 100% and including the pension paid out at death.
Let us suppose, as Yntema (1962) did, that an insurance policy holder can go from degree S_i to degree S_j with a probability p_{ij}. This strong assumption leads to the construction of a Markov chain model in which the $m \times m$ matrix:

$$\mathbf{P} = [\, p_{ij} \,] \tag{9.112}$$

is the transition matrix related to the degree of disability.
For individuals starting at time 0 with S_i as the degree of disability, the mean degree of disability after the nth transition is:

$$\overline{S}_i(n) = \sum_{j=1}^{m} p_{ij}^{(n)} S_j. \tag{9.113}$$

To study the financial equilibrium of the funds, we must compute the limiting value of $\overline{S}_i(n)$:

$$\overline{S}_i = \lim_{n\to\infty} \overline{S}_i(n), \tag{9.114}$$

or

$$\overline{S}_i = \lim_{n\to\infty} \sum_{j=1}^{m} p_{ij}^{(n)} S_j. \tag{9.115}$$

This value can be found by applying **Corollary 9.3** for $i = 1, \ldots, m$.

Numerical example
Using real-life data for silicosis, Yntema (1962) began with the following intermediate degrees of disability:

$S_1 = 10\%$
$S_2 = 30\%$
$S_3 = 50\%$
$S_4 = 70\%$
$S_5 = 100\%$

Using real observations recorded in the Netherlands, he considered the following transition matrix **P**:

$$\mathbf{P} = \begin{bmatrix} .90 & .10 & 0 & 0 & 0 \\ 0 & .95 & .05 & 0 & 0 \\ 0 & 0 & .90 & .05 & .05 \\ 0 & 0 & 0 & .90 & .10 \\ 0 & 0 & .05 & .05 & .90 \end{bmatrix}; \quad (9.116)$$

the transition graph associated with the matrix (9.116) is given in **Figure 9.4**:
This immediately shows that:

(i) all states are aperiodic,

(ii) the set $\{S_3, S_4, S_5\}$ is an essential class (positive recurrent),

(iii) the singleton $\{1\}$ and $\{2\}$ are two inessential transient classes.

Hence a uni-reducible Markov chain can be associated with matrix **P**. We can thus apply **Corollary 9.2**. It follows from relation (9.116) that:

$$\overline{S_i} = \lim_{n\to\infty} \sum_{j=3}^{5} \pi_j S_j \,, \quad (9.117)$$

where (π_3, π_4, π_5) is the unique solution of the linear system:

$$\begin{aligned} \pi_3 &= .9\cdot\pi_3 + 0\cdot\pi_4 + .05\cdot\pi_5, \\ \pi_5 &= .05\cdot\pi_3 + .9\cdot\pi_4 + .05\cdot\pi_5, \\ \pi_4 &= .05\cdot\pi_3 + .05\cdot\pi_4 + .9\cdot\pi_5, \\ 1 &= \pi_3 + \pi_4 + \pi_5. \end{aligned} \quad (9.118)$$

The solution is:

$$\pi_3 = \frac{2}{9}, \ \pi_4 = \frac{3}{9}, \ \pi_5 = \frac{4}{9}. \quad (9.119)$$

Therefore:

$$\overline{S_i} = \left(\frac{2}{9}50 + \frac{3}{9}70 + \frac{4}{9}100\right)\% \quad (9.120)$$

or

$$\overline{S_i} = 79\% \quad (9.121)$$

which is the result obtained by Yntema.

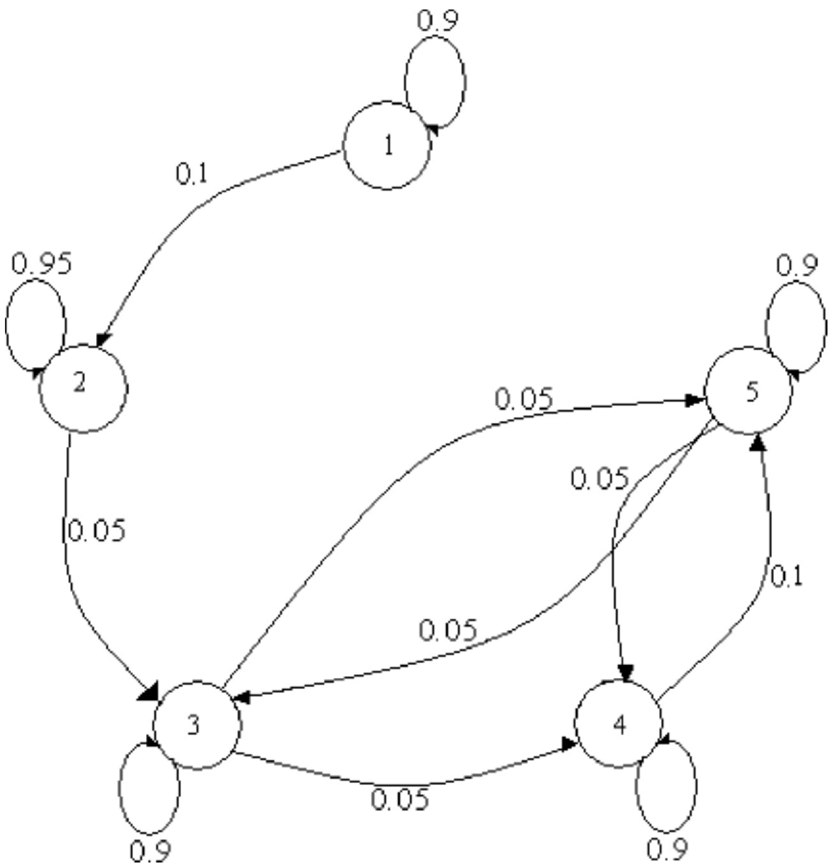

Figure 9.4

The last result proves that the mean degree of disability is, at the limit, independent of the initial state *i*.

Chapter 3

MARKOV RENEWAL PROCESSES, SEMI-MARKOV PROCESSES AND MARKOV RANDOM WALKS

In this chapter, the reader will find the main definitions and results on Markov renewal processes, semi-Markov processes and Markov random walks useful for understanding of the main applications in finance, insurance and reliability developed in the next chapters. A full presentation including the proofs of the theorems recalled here can be found in Janssen-Manca (2006) (chapter 4 to chapter 6).

1 POSITIVE (J-X) PROCESSES

Let us consider a *physical* or *economic system* called S with m possible states, m being a finite natural number.
For simplicity, we will note by I the set of all possible states:

$$I = \{1,...,m\} \tag{1.1}$$

as we already did in Chapter 2 partially devoted to Markov chains.
At time 0, the system S starts from an initial state represented by the r.v. J_0, stays a non-negative random length of time X_1 in this state, and then goes into another state J_1 for a non-negative length of time X_2 before going into J_2, and so on.
So we have a two-dimensional stochastic process in discrete time called a *positive* (J-X) *process*:

$$(J-X) = ((J_n, X_n), n \geq 0) \tag{1.2}$$

supposing

$$X_0 = 0, a.s. \tag{1.3}$$

where the sequence $(J_n, n \geq 0)$ gives the successive *states* of S in time and the sequence $(X_n, n \geq 0)$ gives the successive *sojourn times*.
More precisely, X_n is the time spent by S in state J_{n-1} ($n>0$).
Times at which transitions occur are given by the sequence $(T_n, n \geq 0)$ where:

$$T_0 = 0, T_1 = X_1, ..., T_n = \sum_{r=1}^{n} X_r \tag{1.4}$$

and so

$$X_n = T_n - T_{n-1}, n \geq 1 . \tag{1.5}$$

2 SEMI-MARKOV AND EXTENDED SEMI-MARKOV CHAINS

On the complete probability space $(\Omega,\Im,P)$, the stochastic dynamic evolution of the considered (*J-X*) process will be determined by the following assumptions:
$P(X_0=0)=1$, a.s.,

$$P(J_0=i)=p_i,\ i=1,...,m \text{ with } \sum_{i=1}^{m} p_i = 1, \tag{2.1}$$

for all $n>0, j=1,...,m,$ we have:

$$P(J_n = j, X_n \le x \big| (J_k, X_k), k = 0,...,n-1) = Q_{J_{n-1}j}(x), a.s. \tag{2.2}$$

where any function Q_{ij} $(i,j=1,...,m)$ is a non-decreasing real function null on $\mathbb{R}^+$ such that if

$$p_{ij} = \lim_{x\to+\infty} Q_{ij}(x),\ i,j \in I, \tag{2.3}$$

then:

$$\sum_{j=1}^{m} p_{ij} = 1,\ i \in I. \tag{2.4}$$

With matrix notation, we will write:

$$\mathbf{Q} = \left[Q_{ij}\right],\ \mathbf{P} = \left[p_{ij}\right](= \mathbf{Q}(\infty)),\ \mathbf{p} = (p_1,..., p_m). \tag{2.5}$$

This leads to the following definitions.

Definition 2.1 *Every matrix* $m \times m$ $\mathbf{Q}$ *of non-decreasing functions null on* $\mathbb{R}^+$ *satisfying properties* (2.3) *and* (2.4) *is called a semi-Markov matrix or a semi-Markov kernel.*

Definition 2.2 *Every couple* (**p,Q**) *where* **Q** *is a semi-Markov kernel and* **p** *a vector of initial probabilities defines a positive* (*J,X*) *process*
(*J,X*) $=((J_n,X_n), n \ge 0)$ *with* $I \times \mathbb{R}^+$
as state space, also called a semi-Markov chain (*in short a SMC*)**.**
Sometimes, it is useful that the random variables $X_n, n \ge 0$ take their values in $\mathbb{R}$ instead of $\mathbb{R}^+$, in which case we need the next two definitions.

Definition 2.3 *Every matrix* $m \times m$ $\mathbf{Q}$ *of non-decreasing functions satisfying properties* (2.3) *and* (2.4) *is called an extended semi-Markov matrix or an extended semi-Markov kernel.*

Definition 2.4 *Every couple* (**p,Q**) *where* **Q** *is an extended semi-Markov kernel and* **p** *a vector of initial probabilities defines a* (*J,X*) *process* (*J,X*) **=**$((J_n,X_n),$

$n \geq 0$) *with* $I \times \mathbb{R}$ *as state space, also called an extended semi-Markov chain (in short an ESMC).*

Let us come back to the main condition (2.2); its meaning is clear. For example let us suppose that we observe for a certain fixed n that $J_{n-1}=i$, then the basic relation (2.2) gives us the value of the following conditional probability:

$$P(J_n = j, X_n \leq x \mid (J_k, X_k), k = 0,...,n-1, J_{n-1} = i) = Q_{ij}(x). \qquad (2.6)$$

That is, the knowledge of the value of J_{n-1} suffices to give the conditional probabilistic evolution of the future of the process whatever the values of the other past variables might be.
According to Kingman (1972), the event $\{\omega : J_{n-1}(\omega) = i\}$ is *regenerative* in the sense that the observation of this event gives the complete evolution of the process in the future as it could evolve from n=0 as i the initial state.
(J-X) processes will be fully developed in section 14.

Remark 2.1 The second member of the semi-Markov characterisation property (2.2) does not depend explicitly on n; also we can be precise that we are now studying *homogeneous* semi-Markov chains in comparison with the *non-homogeneous* case where this dependence with respect to n is valid.

3 PRIMARY PROPERTIES

We will start by studying the marginal stochastic processes $(J_n, n \geq 0)$, $(X_n, n \geq 0)$ called respectively the *J-process* and the *X-process*.
(i) The J-process
From the semi-Markov relation (2.2) and Lebesgue's theorem (see Chapter 1, **Proposition 4.1**)), we deduce that a.s.:

$$P(J_n = j \mid (J_k, X_k), k = 0,...,n-1) = Q_{J_{n-1}j}(+\infty). \qquad (3.1)$$

Using the smoothing property (see Chapter 1, **Proposition 6.2**) of conditional expectation, we get

$$P(J_n = j \mid (J_k), k = 0,...,n-1) = E(Q_{J_{n-1}j}(+\infty) \mid (J_k), k = 0,...,n-1), \qquad (3.2)$$

and as the r.v. $Q_{J_{n-1}j}(+\infty)$ is $(J_k, k = 0,...n-1)$,*k*=0,...,*n*-1)-measurable, we finally get from relation (2.3) that:

$$P(J_n = j \mid (J_k), k = 0,...,n-1) = p_{J_{n-1}j}. \qquad (3.3)$$

Since relation (2.4) implies that the matrix **P** is a Markov matrix, we have thus proved the following result.

Proposition 3.1 *The J-process is a homogeneous Markov chain with P as its transition matrix.*

That is the reason why this J-process is called the *imbedded Markov chain* of the considered SMC in which the r.v. J_n represents the state of the system S just after the nth transition.

From results of Chapter 2, **Corollary 9.1**, it follows that in the ergodic case there exists one and only one stationary distribution of probability $\boldsymbol{\pi} = (\pi_1,...,\pi_m)$ satisfying:

$$\begin{aligned} &\pi_i = \sum_{j=1}^{m} \pi_j p_{ji}, j = 1,...,m, \\ &\sum_{i=1}^{m} \pi_i = 1 \end{aligned} \tag{3.4}$$

such that

$$\lim_{n\to\infty} P(J_n = j \mid J_0 = i)(= \lim_{n\to\infty} p_{ij}^{(n)}) = \pi_j, i, j \in I, \tag{3.5}$$

where we know from Chapter 2, relation (9.17) that

$$\left[p_{ij}^{(n)} \right] = \mathbf{P}^n . \tag{3.6}$$

(ii) The X-process

Here, the situation is entirely different because the distribution of X_n depends on J_{n-1}. Nevertheless, we have an interesting property of *conditional independence,* but before giving this property we must introduce some definitions.

Definition 3.1 *The two conditional probability distributions*

$$\begin{aligned} &F_{J_{n-1}J_n}(x) = P(X_n \le x \mid J_{n-1}, J_n), \\ &H_{J_{n-1}}(x) = P(X_n \le x \mid J_{n-1}) \end{aligned} \tag{3.7}$$

are respectively called the conditional and unconditional distributions of the sojourn time X_n.

From the general properties of conditioning recalled in Chapter 1, section 6.2, we successively get

$$\begin{aligned} F_{J_{n-1}J_n}(x) &= E\left(P(X_n \le x \mid (J_k, X_k), k \le n-1, J_n) \mid J_{n-1}, J_n \right) \\ &= E\left(\frac{Q_{J_{n-1}J_n}(x)}{p_{J_{n-1}J_n}} \mid J_{n-1}, J_n \right) \\ &= \frac{Q_{J_{n-1}J_n}(x)}{p_{J_{n-1}J_n}}, \end{aligned} \tag{3.8}$$

provided that $p_{J_{n-1}J_n}$ is strictly positive. If not, we can arbitrarily give to (3.8) for example the value $U_1(x)$ defined as

$$U_1(x)=\begin{cases}0, x<0,\\ 1, x\geq 0.\end{cases} \tag{3.9}$$

Moreover, from the smoothing property, we also have:

$$\begin{aligned} H_{J_{n-1}}(x)(=P(X_n \leq x \mid J_{n-1})) &= E(F_{J_{n-1}J_n}(x) \mid J_{n-1}) \\ &= \sum_{j=1}^{m} p_{J_{n-1}J_n} F_{J_{n-1}J_n}(x). \end{aligned} \tag{3.10}$$

We have thus proved the following proposition.

Proposition 3.2 *As a function of the semi-kernel* **Q,** *the expressions conditional and unconditional distributions of the sojourn time* X_n *are given by:*

$$\begin{aligned} &F_{ij}(x)(=P(X_n \leq x \mid J_{n-1}=i, J_n=j)) = \begin{cases}\dfrac{Q_{ij}(x)}{p_{ij}}, p_{ij}>0,\\ U_1(x), p_{ij}=0,\end{cases} \\ &H_i(x)(=P(X_n \leq x \mid J_{n-1}=i)) = \sum_{j=1}^{m} p_{ij} F_{ij}(x). \end{aligned} \tag{3.11}$$

Remark 3.1

(i) From the last relation (3.11), we can also express the kernel **Q** as a function of the F_{ij}, $i,j=1,\ldots,m$:

$$Q_{ij}(x)=p_{ij}F_{ij}(x), i,j \in I, x \in \mathbb{R}^+. \tag{3.12}$$

So, every SMC can also be characterised by the triple (**p**,**P**,**F**) instead of the couple (**p**,**Q**) where the $m \times m$ matrix **F** is defined as $\mathbf{F}=\left[F_{ij}\right]$ and the functions $F_{ij}, i,j=1,\ldots,m$ are distribution functions on support $\mathbb{R}^+$.

(ii) We can also introduce the *means* related to these conditional and unconditional distribution functions.

When they exist we will write

$$\begin{aligned} \beta_{ij} &= \int_R x dF_{ij}(x), i,j=1,\ldots,m, \\ \eta_i &= \int_R x dH_i(x), i=1,\ldots,m \end{aligned} \tag{3.13}$$

and the last relation (3.11) leading to the relation:

$$\eta_i=\sum_{j=1}^{m} p_{ij}\beta_{ij} . \tag{3.14}$$

The quantities β_{ij}, $i,j=1,\ldots,m$ and $\eta_i, i=1,\ldots,m$ are respectively called the *conditional* and *unconditional means* of the sojourn times.

We can now give the property of *conditional independence*.

Proposition 3.3 *For each integer k, let $n_1,n_2,...,n_k$ be k positive integers such that $n_1<n_2<\cdots<n_k$ and $x_{n_1},...,x_{n_k}$ are k real numbers. Then we have:*

$$\begin{aligned} &P\left(X_{n_1} \le x_{n_1},...,X_{n_k} \le x_{n_k} \middle| J_{n_1-1}, J_{n_1},...,J_{n_k-1}, J_{n_k}\right) \\ &= F_{J_{n_1-1}J_{n_1}}(x_{n_1})...F_{J_{n_k-1}J_{n_k}}(x_{n_k}), \end{aligned} \tag{3.15}$$

that is the k random variables $X_{n_1},...,X_{n_k}$ are conditionally independent given $J_{n_1-1}, J_{n_1},...,J_{n_k-1}, J_{n_k}$.

(i) The *T*-process

By relation (1.4), the sequence $(T_n, n \ge 0)$ represents successive *renewal epochs*, that is, times at which transitions occur.

By analogy with renewal theory, we have the following definition.

Definition 3.2 *The two-dimensional process $((J_n,T_n), n \ge 0)$ is called the Markov renewal process of kernel* **Q.**

Before giving an expression for the marginal distribution of the random vector (J_n,T_n) with values in $I \times \mathbb{R}^+$, given that $J_0=i$, let us define the *marginal distributions* of the (J,T) process $((J_n,T_n), n \ge 0)$:

$$Q_{ij}^n(t) = P(J_n = j, T_n \le t | J_0 = i), i, j \in I, n \ge 0, t \ge 0. \tag{3.16}$$

With $\mathbf{A} = \left[A_{ij}\right]$ and $\mathbf{B} = \left[B_{ij}\right]$, two $m \times m$ matrices of integrable functions, we associate a new matrix $\mathbf{A} \bullet \mathbf{B}$ whose general element $(\mathbf{A} \bullet \mathbf{B})_{ij}$ is the function of *t* defined by:

$$(\mathbf{A} \bullet \mathbf{B})_{ij}(t) = \sum_{k=1}^{m} \int_{\mathbb{R}} A_{kj}(t-y) dB_{ik}(y). \tag{3.17}$$

It can be easily seen that this type of product, called the *convolution product for matrices*, is *associative* but not always commutative.

In the particular case of $A=B$, we set:

$$\begin{aligned} &\mathbf{A} \bullet \mathbf{A} = \mathbf{A}^{(2)},...,\mathbf{A} \bullet \cdots \bullet \mathbf{A} = \mathbf{A}^{(n)} \left(= \left[A_{ij}^{(n)}\right]\right), \\ &\mathbf{A}^{(0)} = (\delta_{ij} U_0), \mathbf{A}^{(1)} = \mathbf{A}. \end{aligned} \tag{3.18}$$

If all the functions $A_{ij}, B_{ij}, i, j = 1,...,m,$ vanish at $-\infty$, we can also use an integration by parts to express (3.17) as follows:

$$(\mathbf{A} \bullet \mathbf{B})_{ij}(t) = \sum_{k=1}^{m} \int_{\mathbb{R}} B_{ik}(t-y) dA_{kj}(y) \tag{3.19}$$

and moreover if **A**=**B**, we get:

$$(\mathbf{A} \bullet \mathbf{B})_{ij}(t) = \sum_{k=1}^{m} \int_{\mathbb{R}} A_{ik}(t-y) dA_{kj}(y) . \tag{3.20}$$

Proposition 3.4 *For all* $n \geq 0$*, we have:*

$$Q_{ij}^{n} = Q_{ij}^{(n)} . \tag{3.21}$$

Moreover, we also have:

$$\lim_{t \to \infty} Q^{(n)}(t) = P^{n} . \tag{3.22}$$

Remark 3.2 It is clear that the above properties proved for SMC are also valid for ESMC. The only difference is that the T-process can no longer be interpreted as an extension of a classical renewal process but in fact as a random walk, as the r.v. T_n then take their values in $\mathbb{R}$ and no longer in $\mathbb{R}^+$ (see section 14).

4 EXAMPLES

Semi-Markov theory is one of the most productive subjects of stochastic processes to generate applications in real-life problems, particularly in the following fields: Economics, Manpower models, Insurance, Finance (more recently), Reliability, Simulation, Queuing, Branching processes, Medicine (including survival data), Social Sciences, Language Modelling, Seismic Risk Analysis, Biology, Computer Science, Chromatography and Fluid mechanics.
Important results in such fields may be found in Janssen (1986) and Janssen and Limnios (1999).
Let us give three examples in the fields of insurance and reliability.

Example 4.1: *The claim process in insurance*
Let us consider an insurance company covering *m* types of risks or having *m* different types of customers for the same risk forming the set $I=\{1,...,m\}$.
For example, in automobile insurance, we can distinguish three types of drivers: *good, average* and *bad* and so *I* is a space consisting of three states: 1 for good, 2 for average and 3 for bad.
Now, let $(X_n, n \geq 1)$ represent the sequence of successive observed *claim amounts*, $(Y_n, n \geq 1)$ the sequence of interarrivals between two successive claims and $(J_n, n \geq 1)$ successive *types of observed risks*.
In the classical model of risk theory called the Cramer-Lundberg model (1909, 1955), it is supposed that there is only one type of risk and the claim arrival process is a *Poisson process* of parameter λ; later, Andersen (1967) extends this model to an arbitrary *renewal process* and moreover in these two classical models, the process of claim amounts is a renewal process independent of the claim arrival process.

The consideration of an SMC for the two-dimensional processes $((J_n, X_n), n \geq 0)$ or/and $((J_n, Y_n), n \geq 0)$ gives the possibility to introduce a certain dependence between the successive claim amounts. This model was first developed by Janssen (1969b, 1970, 1977) along the lines of Miller's work (1962) and since then has lead to a lot of extensions, see for example Asmussen (2000).

Example 4.2: *Occupational illness insurance*
This problem is related to occupational illness insurance with the possibility of developing *partial* or *permanent* disability. In this case the amount of the incapacitation allowance depends on the *degree of disability* recognised for the policyholder by the occupational health doctor, in general on a yearly basis, because this degree is a function of a professional illness which can become better or worse.
Considering as in the example Chapter 2, section 9.7, this invalidity degree as a stochastic process $(J_n, n \geq 0)$ where J_n represents the value of this degree when the illness really takes its course, we must then introduce the r.v. X_n representing the time between two successive transitions from J_{n-1} to J_n.
In practice, these transitions can be observed with periodic medical inspections.
The assumption that the *J-X* process is an SMC extends the Markov model of Chapter 2 and is fully treated in Janssen and Manca (2006).

Example 4.3: *Reliability*
There are many examples of semi-Markov models in reliability theory, see for example Osaki (1985) and more recently in Limnios and Oprişan (2001), (2003).
Let us consider a so-called *reliability system* S that can be at any time t in one of the m states of $I=\{1,\ldots,m\}$.
The stochastic process of the successive states of S is represented by $S=(S_t, t \geq 0)$.
The state space I is partitioned into two sets U and D so that

$$I = U \bigcup D,\ U \bigcap D = \varnothing,\ U \neq \varnothing,\ D \neq \varnothing. \tag{4.1}$$

The interpretation of these two sets is the following : the subset U contains all "good" states, in which the system is working and the subset D of all "bad " states in which the system is not working well or has failed.
The indicators used in reliability theory are the following ones:
(i) the *reliability function* R giving the probability that the system was always working from time 0 to time t:

$$R(t) = P\left(S_u \in U, \forall u \in [0,t]\right), \tag{4.2}$$

(ii) the *pointwise availability function* A giving the probability that the system is working at time t whatever happens on $(0,t]$:

$$A(t) = P\left(S_t \in U\right), \tag{4.3}$$

(iii) the *maintainability function M* giving the probability that the system, being in D on $[0,t)$, will leave the set D at time t:

$$M(t) = P\left(S_u \in D, u \in [0,t), S_t \in U\right). \tag{4.4}$$

5 MARKOV RENEWAL PROCESSES, SEMI-MARKOV AND ASSOCIATED COUNTING PROCESSES

Let us consider an SMC of kernel **Q**; we then have the following definitions.

Definition 5.1 *The two-dimensional process* $(J,T)=((J_n,T_n),n\geq 0)$ *where* T_n *is given by relation* (1.4) *is called a Markov renewal sequence or Markov renewal process.*

Çinlar (1969) also gives the term *Markov additive process*. It is justified by the fact that, using relation (1.5), we get:

$$\begin{aligned} &P(J_{n+1} = j, T_{n+1} \leq x \mid (J_k, T_k), k = 0,...,n) \\ &= P(J_{n+1} = j, X_{n+1} \leq x - T_n \mid (J_k, T_k), k = 0,...,n) = \mathrm{Q}_{J_n j}(x - T_n). \end{aligned} \tag{5.1}$$

This last equality shows that the (J,T) process is a Markov process with $I \times \mathbb{R}^+$ as state space and having the "additive property":

$$T_{n+1} = T_n + X_{n+1}. \tag{5.2}$$

Let us say that according to the main definitions of Chapter 2, **Definition 2.1**, always in the case of positive (J,X) chains, the random variables $T_n,(n\geq 0)$ are from now on called *Markov renewal times* or simply *renewal times*, the random variables $X_n,(n\geq 1)$ *interarrival* or *sojourn times* and the random variables $J_n,(n\geq 0)$ the *state variables*.

We will now define the *counting processes* associated with any Markov renewal process (in short MRP) as we already did in the special case of renewal theory.

For any fixed time t, the r.v. $N(t)$ represents the *total number of jumps or transitions of the* (J,X) *process on* $(0,t]$, including possible transitions from any state towards itself (virtual transitions), transitions supposed to be observable.

As in renewal theory, we have:

$$N(t) > t \Leftrightarrow T_n \leq t. \tag{5.3}$$

But here, we can be more precise and only count the total number of passages in a fixed state I always in $(0,t]$ represented by the r.v. $N_i(t)$.

Clearly, we can write:

$$N(t) = \sum_{i=1}^{m} N_i(t), t \geq 0. \tag{5.4}$$

Definition 5.2 *To each Markov renewal process, the following* $m+1$ *stochastic processes are associated respectively with values in* $\mathbb{N}$ *:*

(i) *the N-process* $(N(t), t \geq 0)$,

(ii) *the* N_i*-process* $(N_i(t), t \geq 0)$, $i=1,...,m,$

respectively called the associated total counting process and the associated partial counting processes with of course:

$$N(0)=0, N_i(0)=0, i=1,...,m. \tag{5.5}$$

It is now easy to introduce the notion of a *semi-Markov process* by considering at time t, the state entered at the last transition before or at t, that is $J_{N(t)}$.

Definition 5.3 *With each Markov renewal process, we associate the following stochastic Z-process with values in I:*

$$Z=(Z(t), t \geq 0), \tag{5.6}$$

with:

$$Z(t)=J_{N(t)}. \tag{5.7}$$

This process will be called the associated semi-Markov process or simply the semi-Markov process (*in short SMP*) *of kernel* **Q**.

Remarks 5.1

1) As in renewal theory, we will often use counting variables including the initial renewal, that is:

$$\begin{aligned} N'(t) &= N(t)+1, \\ N_i^{'}(t) &= N_i(t)+\delta_{iJ_0}. \end{aligned} \tag{5.8}$$

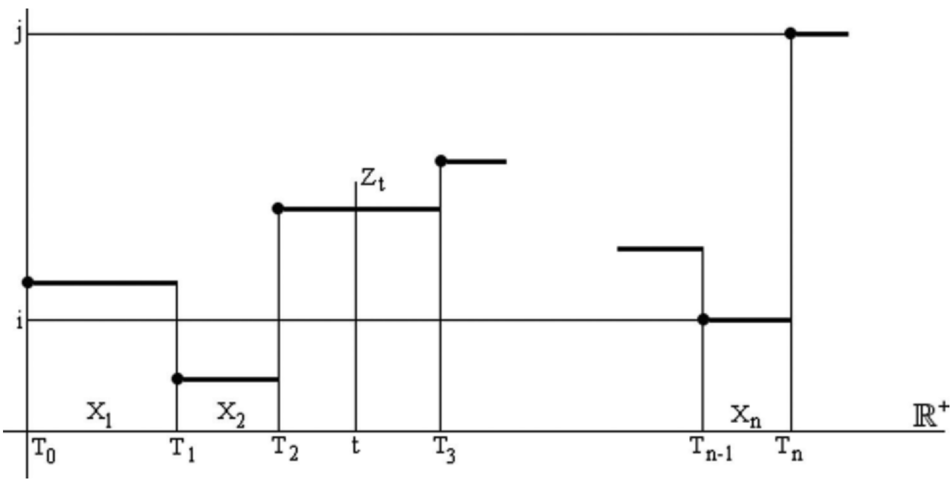

Figure 5.1: a trajectory of an SMP

2) **Figure 5.1** gives a typical trajectory of MRP and SMP.

3) It is now clear that we can immediately consider an MRP defined by kernel **Q** without speaking explicitly of the basic (*J,X*) process with the same kernel **Q**, because the basic property (2.6) is equivalent to (5.1).

6 MARKOV RENEWAL FUNCTIONS

Let us consider an MRP of kernel **Q** and to avoid trivialities, we will assume that:

$$\sup_{i,j} Q_{ij}(0) < 1, \tag{6.1}$$

where the functions Q_{ij} are defined by relation (2.2).

If the initial state J_0 is *i*, let us define the r.v. $T_n\left(i|i\right), n \geq 1$, as the times (possibly infinite) of *successive returns* to state *i*, also called *successive entrance times* into $\{i\}$.

From the regenerative property of RRP, whenever the process enters into state *i*, say at time *t*, the evolution of the process on $\left[t,\infty\right)$ is probabilistically the same as if we had started at time 0 in the same state *i*.

It follows that the process $\left(T_n\left(i|i\right), n \geq 0\right)$ with:

$$T_0\left(i|i\right) = 0 \tag{6.2}$$

is a renewal process that could be possibly defective.

From now on, the r.v. $T_n\left(i|i\right)$ will be called the *nth return time to state i*.

More generally, let us also fix state *j*, different from state *i* already fixed; we can also define the *nth return* or *entrance time to state j*, but *starting from i as initial state*. This time, possibly infinite too, will be represented by $\left(T_n\left(j|i\right), n \geq 0\right)$, using here too the convention that

$$T_0\left(j|i\right) = 0\,. \tag{6.3}$$

Now, the sequence $\left(T_n\left(j|i\right), n \geq 0\right)$ is a delayed renewal process with values in $\mathbb{R}^+$.

It is thus defined by two d.f.: G_{ij} being that of $T_1\left(j|i\right)$ and G_{jj} that of $T_2\left(j|i\right) - T_1\left(j|i\right)$, so that:

$$\begin{aligned} G_{ij}(t) &= P\left(T_1\left(j|i\right) \leq t\right), \\ G_{jj}(t) &= P\left(T_n\left(j|i\right) - T_{n-1}\left(j|i\right) \leq t\right), \; n \geq 2. \end{aligned} \tag{6.4}$$

Of course, the d.f. G_{jj} suffices to define the renewal process $\left(T_n\left(j|j\right), \, n \geq 0\right)$.

Remark 6.1 From the preceding definitions, we can also write that:

$$\begin{aligned} &G_{ij}(t) = P\left(N_j(t) > 0 \middle| J_0 = i\right);\ i, j \in I, \\ &P\left(T_1\left(j \middle| i\right) = +\infty\right) = 1 - G_{ij}(+\infty) \end{aligned} \tag{6.5}$$

and for the mean of the $T_n\left(i \middle| i\right), n \geq 1$, possibly infinite, we get:

$$\mu_{ij} = E\left(T_1\left(j \middle| i\right)\right) = \int_0^\infty t dG_{ij}(t), \tag{6.6}$$

with the usual convention that

$$0 \cdot (+\infty) = 0. \tag{6.7}$$

The means $\mu_{ij}, i, j \in I$ are called the *first entrance* or *average return times*.

Lemma 6.1 *The functions* $G_{ij}, i, j \in I$ *satisfy the following relationships:*

$$G_{ij}(t) = \sum_{k=1}^{m} G_{kj} \bullet Q_{ik}(t) + (1 - G_{jj}) \bullet Q_{ij}(t), i, j \in I, t \geq 0. \tag{6.8}$$

For each possibly delayed renewal process defined by the couple (G_{ij}, G_{jj}), *i,j* belonging to *I*, we will represent by A_{ij} and R_{ij} the associated renewal functions defined by relations 2(2.4) and 2(3.12) so that:

$$\begin{aligned} &A_{ij}(t) = E(N_j(t) \middle| J_0 = i), \\ &R_{ij}(t) = E(N_j'(t) \middle| J_0 = i) \end{aligned} \tag{6.9}$$

and by relations (5.8):

$$R_{ij}(t) = \delta_{ij} U_0(t) + A_{ij}(t). \tag{6.10}$$

From relations 2(9.7), 2(3.9) and 2(3.14), we get:

$$\begin{aligned} &R_{jj}(t) = \sum_{n=0}^{\infty} G_{jj}^{(n)}(t), j \in I, \\ &R_{ij}(t) = G_{ij} \bullet R_{jj}(t). \end{aligned} \tag{6.11}$$

Or equivalently, we have:

$$R_{ij}(t) = \delta_{ij} U_0(t) + G_{ij} \bullet \sum_{n=0}^{\infty} G_{jj}^{(n)}(t), i, j \in I. \tag{6.12}$$

Proposition 6.1 *Assumption* $m < \infty$ *implies that:*

(i) *at least one of the renewal processes* $\left(T_n\left(j \middle| j\right), n \geq 0\right), j \in I$ *is not defective,*

(ii) *for all i belonging to I, there exists a state s such that*

$$\lim_n T_n\left(s \middle| i\right) = +\infty,\ a.s., \tag{6.13}$$

(iii) *for the r.v.* T_n *defined by relation* (1.4)*, given that* $J_0 = i$ *whatever i is, we have a.s. that*

$$\lim_n T_n = +\infty. \tag{6.14}$$

The following relations will express the renewal functions $R_{ij},\ i,j \in I$ as a function of the kernel **Q** instead of the m^2 functions G_{ij}.

Proposition 6.2 *For every i and j of I, we have that:*

$$R_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t). \tag{6.15}$$

Using matrix notation with:

$$\mathbf{R} = \left[R_{ij} \right], \tag{6.16}$$

relation (6.14) takes the form:

$$\mathbf{R} = \sum_{n=0}^{\infty} \mathbf{Q}^{(n)}. \tag{6.17}$$

Let us now introduce the *L-S transform of matrices.*
For any matrix of suitable functions A_{ij} from $\mathbb{R}^+$ to $\mathbb{R}$ represented by

$$\mathbf{A} = \left[A_{ij} \right] \tag{6.18}$$

we will represent its L-S transform by:

$$\bar{\mathbf{A}} = \left[\bar{A}_{ij} \right] \tag{6.19}$$

with

$$\bar{A}_{ij}(s) = \int_0^{\infty} e^{-st} dA_{ij}(t). \tag{6.20}$$

Doing so for the matrix $\mathbf{R}$, we get the matrix form of relation (6.15),

$$\bar{\mathbf{R}}(s) = \sum_{n=0}^{\infty} \left(\bar{\mathbf{Q}}(s) \right)^n . \tag{6.21}$$

From this last relation, a simple algebraic argument shows that, for any $s>0$, relations

$$\bar{\mathbf{R}}(s)(\mathbf{I} - \bar{\mathbf{Q}}(s)) = (\mathbf{I} - \bar{\mathbf{Q}}(s)\bar{\mathbf{R}}(s) = \mathbf{I} \tag{6.22}$$

hold and so, we also have that:

$$\bar{\mathbf{R}}(s) = (\mathbf{I} - \bar{\mathbf{Q}}(s))^{-1}. \tag{6.23}$$

We have thus proved the following proposition.

Proposition 6.3 *The Markov renewal matrix* $\mathbf{R}$ *is given by*

$$\mathbf{R} = \sum_{n=0}^{\infty} \mathbf{Q}^{(n)}, \tag{6.24}$$

the series being convergent in $\mathbb{R}^+$.
Moreover, the L-S transform of the matrix $\mathbf{R}$ *has the form:*

$$\overline{\mathbf{R}} = (\mathbf{I} - \overline{\mathbf{Q}})^{-1}, \tag{6.25}$$

the inverse existing for all positive s.

The knowledge of the Markov renewal matrix $\mathbf{R}$ or its L-S transform $\overline{\mathbf{R}}$ leads to useful expressions for d.f. of the first entrance times.

Proposition 6.4 *For the L-S transforms of the first entrance time distributions, we have:*

$$\overline{G}_{ij}(s) = \begin{cases} \overline{R}_{ij}(s)(\overline{R}_{jj}(s))^{-1}, i \neq j, \\ 1 - (\overline{R}_{jj}(s))^{-1}, i = j. \end{cases} \tag{6.26}$$

Inversely, we have:

$$\overline{R}_{ij}(s) = \begin{cases} \dfrac{\overline{G}_{ij}(s)}{1 - \overline{G}_{jj}(s)}, i \neq j, \\ 1 - (\overline{G}_{jj}(s))^{-1}, i = j. \end{cases} \tag{6.27}$$

7 CLASSIFICATION OF THE STATES OF AN MRP

To give the classification of the states here, we will proceed as we did in the case of Markov chains: that is, by considering the embedded renewal processes or delayed renewal processes of return times in the different states of *I*.
This gives the following definition.

Definition 7.1 *The state j of I is said to be recurrent, transient, aperiodic or periodic with period d. according to the associated embedded renewal processs is recurrent, transient,r aperiodic or periodic with period d.*
Moreover, j is positive (*or non-null*) *recurrent iff* μ_{jj} *is finite.*

The next proposition establishes the interaction between classification of the states of an MRP and that of the same states but for the imbedded MC $(J_n, n \geq 0)$.

Proposition 7.1
(i) *j is recurrent for the MRP iff j is recurrent, necessarily positive in the imbedded MC,*
(ii) *if* $\sup_{i,j} b_{ij} < \infty$*, then j recurrent for the MRP implies that it is positive recurrent,*
(iii) *j transient for the MRP implies that j is also transient for the imbedded MC.*

Definition 7.2 *State j of an MRP is accessible from state i, or j can be reached from state i, if there exists a strictly positive t such that:*

$$P\left(Z(t)=j \middle| Z(0)=i\right)>0\,. \tag{7.1}$$

Remark 7.1 A state j is periodic for the MRP iff the d.f. G_{jj} is arithmetic.

It is clear that the periodicity of a state j for the MRP bears no relation to the periodicity of this state in the embedded MC.
For the periodicity in the MRP, Çinlar (1975b) had nevertheless proved the following result:
If j can be reached from i and if i can be reached from j, then state i and j are both aperiodic or both periodic and, in this latter case, have the same period.

Definition 7.3 *An MRP will be called irreducible if every state can be reached from any state.*

From **Remark 7.1**, it follows that in the irreducible case, all the states are both periodic with the same period or both aperiodic.
From **Proposition 7.1**, we can deduce the following result.

Proposition 7.2 *An MRP is irreducible iff the imbedded MC is also irreducible.*

Definition 7.4 *An MRP will be called ergodic iff it is irreducible aperiodic, and if the imbedded MC is also aperiodic.*
The semi-Markov kernel **Q** *corresponding to an ergodic MRP is also called an ergodic kernel.*

From **Remark 7.1**, the ergodicity of an MRP implies that of the imbedded MC, but the ergodicity of the imbedded MC only implies the irreducibility of the MRP.

8 THE MARKOV RENEWAL EQUATION

This paragraph will extend the basic results related to the renewal equation developed in section 4 of Chapter 2 to the Markov renewal case.
Let us consider an MRP of kernel **Q**.
From relation (6.15), we get:

$$\begin{aligned} R_{ij}(t) &= \delta_{ij}U_0(t)+\sum_{n=1}^{\infty}Q_{ij}^{(n)}(t) \\ &= \delta_{ij}U_0(t)+\left(Q\bullet R\right)_{ij}(t). \end{aligned} \tag{8.1}$$

Using matrix notation with:

$$\mathbf{I}(t)=\left[\delta_{ij}U_0(t)\right], \tag{8.2}$$

relations (8.1) take the form:

$$\mathbf{R}(t)=\mathbf{I}(t)+\mathbf{Q}\bullet\mathbf{R}(t). \tag{8.3}$$

This integral matrix equation is called the *Markov renewal equation* for **R**.
To obtain the corresponding matrix integral equation for the matrix

$$\mathbf{H}=\left[H_{ij}\right], \tag{8.4}$$

we know, from relation (6.10) that

$$\mathbf{R}(t)=\mathbf{I}(t)+\mathbf{H}(t). \tag{8.5}$$

Inserting this expression of $\mathbf{R}(t)$ in relation (8.3), one obtains:

$$\mathbf{H}(t)=\mathbf{Q}(t)+\mathbf{Q}\bullet\mathbf{H}(t) \tag{8.6}$$

which is the *Markov renewal equation for* **H**.
Of course, for m=1, this last equation gives the classical renewal equation (4.1) of Chapter 2.
In fact, the Markov renewal equation (8.3) is a particular case of the matrix integral equation of the type:

$$\mathbf{f}=\mathbf{g}+\mathbf{Q}\bullet\mathbf{f}, \tag{8.7}$$

called an integral equation of *Markov renewal type* (in short MRT), where

$$\mathbf{f}=\left(f_1,...,f_m\right)',\mathbf{g}=\left(g_1,...,g_m\right)' \tag{8.8}$$

are two column vectors of functions having all their components in B, the set of single-variable measurable functions, bounded on finite intervals or to B^+ if all their components are non-negative.

Proposition 8.1 *The Markov integral equation of MRT,*

$$\mathbf{f}=\mathbf{g}+\mathbf{Q}\bullet\mathbf{f} \tag{8.9}$$

with **f**,**g** *belonging to* B^+*, has the unique solution:*

$$\mathbf{f}=\mathbf{R}\bullet\mathbf{g}. \tag{8.10}$$

9 ASYMPTOTIC BEHAVIOUR OF AN MRP

We will give asymptotic results, first for the Markov renewal functions and then for solutions to integral equations of an MRT.
To finish, we will apply these results to transition probabilities of an SMP.

9.1 Asymptotic Behaviour Of Markov Renewal Functions

We know that the renewal function R_{ij} , i,j belonging to I, is associated with the delayed renewal process, possibly transient, characterized by the couple (G_{ij},G_{jj}).d.f. on $\mathbb{R}^+$.
Let us recall that μ_{ij} represents the mean, possibly infinite, of the d.f. G_{ij}.

Proposition 9.1 *For all i,j of I, we have:*

(i) $\lim_{t\to\infty}\frac{R_{ij}(t)}{t}=\frac{1}{\mu_{jj}},$ (9.1)

(ii) $\lim_{t\to\infty}\frac{R_{ij}(t)-R_{ij}(t-\tau)}{\tau}=\frac{\tau}{\mu_{jj}}$, for every fixed τ. (9.2)

The next proposition, due to Barlow (1962), is a useful complement to the last proposition as it gives a method for computing the values of the mean return times $\mu_{jj}, j\in I$, in the ergodic case.

Proposition 9.2 *For an ergodic MRP, the mean return times satisfy the following linear system:*

$$\mu_{ij}=\sum_{k\neq j}p_{ik}\mu_{kj}+\eta_i, i=1,...,m. \tag{9.3}$$

In particular, for i=j, we have:

$$\mu_{jj}=\frac{1}{\pi_j}\sum_k \pi_k\eta_k, j=1,...,m, \tag{9.4}$$

where the $\eta_i, i\in I$ *are defined by relation* (3.14)*, and where* $\boldsymbol{\pi}=(\pi_1,...,\pi_m)$ *is the unique stationary distribution of the imbedded Markov chain.*

Remark 9.1 In a similar manner, Barlow (1962) proved that if $\mu_{ij}^{(2)}, i,j\in I$ is the second order moment related to the d.f. G_{ij}, then:

$$\mu_{ij}^{(2)}=\eta_i^{(2)}+\sum_{k\neq j}p_{ik}(\mu_{ik}^{(2)}+2b_{ik}\mu_{kj}) \tag{9.5}$$

and in particular for *i=j*:

$$\mu_{jj}^{(2)}=\frac{1}{\pi_j}(\sum_k \pi_k\eta_k^{(2)}+2\sum_{k\neq j}\sum_l \pi_l p_{lk}b_k\mu_{kj}) \tag{9.6}$$

with

$$\eta_k^{(2)}=\int_{[0,\infty)}x^2dH_k(x), k\in I, \tag{9.7}$$

provided that these quantities are finite.

9.2 Asymptotic Behaviour Of Solutions Of Markov Renewal Equations

Under the assumptions of **Proposition 8.1**, we know that the integral system (8.9), that is

$$f_i(t) = g_i(t) + \sum_j \int_{[0,t]} f_j(t-s) dQ_{ij}(s), i \in I, \tag{9.8}$$

has the unique solution

$$f_i(t) = \sum_j \int_{[0,t]} g_j(t-s) dR_{ij}(s), i \in I. \tag{9.9}$$

For the asymptotic behaviour of this solution for t tending toward $+\infty$, we have the analogue of **Proposition 4.2** of Chapter 2, i.e. the *key renewal theorem*.

Proposition 9.3 (*Key Markov renewal theorem*)
For any ergodic MRP, we have:

$$\lim_{t\to\infty} \sum_j \int_{[0,t]} g_j(t-s) dR_{ij}(s) = \frac{\sum_j \pi_j \int_0^\infty g_j(y) dy}{\sum_j \pi_j \eta_k}, \tag{9.10}$$

provided that the functions g_i, i belonging to I, are directly Riemann integrable.

10 ASYMPTOTIC BEHAVIOUR OF SMP

10.1 Irreducible Case

Let us consider the SMP ($Z(t)$, $t \geq 0$) associated with the MRP of kernel **Q** and defined by relation (5.6).
Starting with $Z(0) = i$, it is important for the applications to know the probability of being in state j at time t, that is:

$$\phi_{ij}(t) = P\left(Z(t) = j \middle| Z(0) = i\right). \tag{10.1}$$

A simple probabilistic argument using the regenerative property of the MRP gives the system satisfied by these probabilities as a function of the kernel **Q**:

$$\phi_{ij}(t) = \delta_{ij}(1 - H_i(t)) + \sum_k \int_0^t \phi_{kj}(t-y) dQ_{ik}(y),\ i, j \in I. \tag{10.2}$$

It is also possible to express the transition probabilities of the SMP with the aid of the first passage time distributions $G_{ij}, i, j \in I$:

$$\phi_{ij}(t) = \phi_{jj} \bullet G_{ij}(t) + \delta_{ij}(1 - H_i(t)),\ i, j \in I. \tag{10.3}$$

If we fix the value j in relations (10.2), we see that the m relations for i=1,...,m form a Markov renewal type equation (in short MRE) of form (8.9).
Applying **Proposition 8.1**, we immediately get the following proposition.

Proposition 10.1 *The matrix of transition probabilities*

$$\mathbf{\Phi} = \left[\phi_{ij}\right] \tag{10.4}$$

is given by

$$\mathbf{\Phi} = \mathbf{R} \bullet (\mathbf{I} - \mathbf{H}) \tag{10.5}$$

with

$$\mathbf{H} = \left[\delta_{ij} H_i\right]. \tag{10.6}$$

So, instead of relation (10.3), we can now write:

$$\phi_{ij}(t) = \int_{[0,t]} (1 - H_j(t-y)) dR_{ij}(y). \tag{10.7}$$

Remark 10.1 *Probabilistic interpretation of relation* (10.7)
This interpretation is analogous to that of the renewal density given in Chapter 2,

Remark 10.2: the "infinitesimal" quantity $dR_{ij}(y)$ ($=r_{ij}(y)dy$, if $r_{ij}(y)$ is the density of the function R_{ij}, if it exists) represents the probability that there is a Markov renewal into state j in the time interval $(y,y+dy)$, starting at time 0 in state i.
Of course, the factor $(1 - H_j(t-y))$ represents the probability of not leaving state j before a time interval of length $t-y$.

The behaviour of transition probabilities of matrix (10.4) will be given in the next proposition.

Proposition 10.2 *Let* $Z = \left(Z(t), t \geq 0\right)$ *be the SMP associated with an ergodic MRP of kernel* **Q***; then:*

$$\lim_{t\to\infty} \phi_{ij}(t) = \frac{\pi_j \eta_j}{\sum_k \pi_k \eta_k}, \quad i, j \in I. \tag{10.8}$$

Remark 10.3
(i) As the limit in relation (10.8) does not depend on i, **Proposition 10.2** establishes an *ergodic property* saying that:

$$\lim_{t\to\infty} \phi_{ij}(t) = \Pi_j,$$
$$\Pi_j = \frac{\pi_j \eta_j}{\sum_k \pi_k \eta_k}. \tag{10.9}$$

(ii) As the number m of states is finite, it is clear that $\left(\Pi_j, j \in I\right)$ is a probability distribution. Moreover, as $\pi_j > 0$ for all j (see relation (9.89) of Chapter 2), we also have

$$\Pi_j > 0, j \in I. \tag{10.10}$$

So, asymptotically, every state is reachable with a strictly positive probability.

(iii) In general, we have:

$$\lim_{n\to\infty} p_{ij}^{(n)} \neq \lim_{t\to\infty} \phi_{ij}(t) \tag{10.11}$$

since of course

$$\pi_j \neq \Pi_j, j \in I. \tag{10.12}$$

This shows that the limiting probabilities for the imbedded Markov chain are not, in general, the same as taking limiting probabilities for the SMP.
From **Propositions 10.2** and **9.2**, we immediately get the following corollary.

Corollary 10.1 *For an ergodic MRP, we have:*

$$\Pi_j = \frac{\pi_j}{\mu_{jj}}. \tag{10.13}$$

This result says that the limiting probability of being in state j for the SMP is the ratio of the mean sojourn time in state j to the mean return time of j.
This intuitive result also shows how the different return times and sojourn times have a crucial role in explaining why we have relation (10.13) as, indeed, for the imbedded MC, these times have no influence.

10.2 Non-Irreducible Case

It happens very often that stochastic models used for applications need non-irreducible MRP, as for example, in presence of an absorbing state, i.e. a state j such that

$$p_{jj} = 1. \tag{10.14}$$

We will now see that the asymptotic behaviour is easily deduced from the irreducible case studied above.

10.2.1 Uni-Reducible Case

As for Markov chains, this is simply the case in which the imbedded MC is uni-reducible so that there exist l ($l<m$) transient states, and so that the other $m-l$ states form a recurrent class C.
We always suppose aperiodicity both for the imbedded MC and the considered MRP.
Let $T=\{1,...,l\}$ be the set of transient states $(T = I - C)$.From **Proposition 9.5** of Chapter 2, we know that:

$$\lim_{t\to\infty}\phi_{ij}(t)=0,\ i,j\in T. \tag{10.15}$$

Moreover, from **Proposition 10.2** and relation (10.3):

$$\lim_{t\to\infty}\phi_{ij}(t)=G_{ij}(\infty)\frac{\pi_j\eta_j}{\sum_{k=l+1}^{m}\pi_k\eta_k},\ i,j\in C, \tag{10.16}$$

where $(\pi_{l+1},...,\pi_m)$ represents the unique stationary probability distribution of the sub-Markov chain with C as state space.
Since

$$G_{ij}(\infty)=f_{ij}, \tag{10.17}$$

we get

$$G_{ij}(\infty)=f_{i,C}, \tag{10.18}$$

where $f_{i,C}$ is the probability that the system, starting in state i, will be absorbing by the recurrent class C.
As there is only one essential class, we know that for all states i of I:

$$f_{i,C}=1, \tag{10.19}$$

proving so the following proposition.

Proposition 10.3 *For any periodic uni-reducible MRP, we have:*

$$\lim_{t\to\infty}\phi_{ij}(t)=\Pi'_j,\ j\in I, \tag{10.20}$$

where

$$\Pi'_j=\begin{cases}0,\ j\in T,\\ \dfrac{\pi_j\eta_j}{\sum_{k=l+1}^{m}\pi_k\eta_k},\ j\in C.\end{cases} \tag{10.21}$$

Here too, as the limit in (10.21) is independent of the initial state i, this result gives an *ergodic property*.

10.2.2. General Case

For any aperiodical MRP, there exists a unique partition of the state space I:

$$I=T\cup C_1\cup\cdots\cup C_r,\ r<m, \tag{10.22}$$

where T represents the set of transient states and $C_\nu, \nu=1,...,r$ represents the ν th essential class necessarily formed of positive recurrent states.
From Chapter 2, we know that the system will finally enter one of the essential classes and will then stay in it forever. So a slight modification of the last proposition leads to the next result.

Proposition 10.4 *For any aperiodic MRP, we have :*

$$\lim_{t\to\infty}\phi_{ij}(t)=\Pi_{ij}^{'},i,j\in I, \tag{10.23}$$

with, for any $j\in C_\nu,\nu=1,...,r$ *:*

$$\Pi_{ij}^{'}=\begin{cases}\Pi_j^{'\nu}, & i\in C_\nu,\nu=1,...,r,\\ 0, & i\in C_{\nu'},\nu\neq\nu',\nu'=1,...,r,\\ f_{i,C_\nu}\Pi_j^{'\nu}, & i\in T\end{cases} \tag{10.24}$$

where $(\Pi_j^{'\nu},j\in C_\nu)$ *is the only stationary distribution of the sub-SMP with* C_ν *as state space, that is :*

$$\Pi_j^{'}=\frac{\pi_j\eta_j}{\sum_{k\in C_\nu}\pi_k\eta_k}, \tag{10.25}$$

where $\left(\pi_k^\nu,k\in C_\nu\right)$ *is the unique stationary distribution of the sub-Markov chain with* C_ν *as state space and* $\left(f_{i,C_\nu},i\in T\right)$ *is the unique solution of the linear system*

$$y_i-\sum_{j\in T}p_{ij}y_j=\sum_{j\in C_\nu}p_{ij},i\in T. \tag{10.26}$$

Note that, in this proposition, the ergodic property is lost; this is due to the presence of the quantities f_{i,C_ν} in relation (10.24).

11 DELAYED AND STATIONARY MRP

Let us suppose we begin to observe the evolution of an economic or physical system S at time T_0 and that the probabilistic evolution of this system is like a semi-Markov process.

There is absolutely no reason that we should observe a transition of the system at time 0. In fact, we observe the state $Z(0)$ while waiting for the first observed transition occurring at random time T_1. The first lifetime X_1 is a residual time and may have its own distribution function.

This leads to the concept of *delayed MRP*.

Definition 11.1 *The bidimensional process* $\left((J_n,T_n),n\geq 0\right)$ *with*

$$\begin{aligned}&T_0=0,\\&T_n=X_1+\cdots+X_n,\quad n\geq 1,\end{aligned} \tag{11.1}$$

is called a delayed Markov renewal sequence or delayed Markov renewal process (*in short DMRP*) *of triplet* $\left(\mathbf{p},\tilde{\mathbf{Q}},\mathbf{Q}\right)$ *if*

$$
\begin{aligned}
&\text{(i)}P(J_0 = i) = p_i, i \in I,\\
&\text{(ii)}P\left(J_1 = j, X_1 \le x \middle| J_0 = i\right) = \tilde{Q}_{ij}(x),\ i, j \in I,\\
&\text{(iii)}P\left(J_n = j, X_n \le x \middle| (J_k, X_k), k = 0,...,n-1\right) = Q_{ij}(x), i, j \in I, x \in \mathbb{R}, n > 1.
\end{aligned}
\tag{11.2}
$$

This definition is clearly based upon the supposition that the m-dimensional vector $\mathbf{p} = (p_1,..., p_m)$ represents a probability distribution on I and that matrices $\tilde{\mathbf{Q}}, \mathbf{Q}$ are two semi-Markov kernels.

As in the case of renewal theory, there exist simple relations between renewal functions, marginal distributions, etc… of a DMRP of triplet $(\mathbf{p}, \tilde{\mathbf{Q}}, \mathbf{Q})$ and the corresponding functions of the classical associated MRP of kernel $\mathbf{Q}$.

So, with the convention of adding a tilde to the functions related to the DMR, let $\tilde{R}_{ij}(t)$ be the Markov renewal functions of the DMRP, that is:

$$
\tilde{R}_{ij}(t) = E\left(\tilde{N}_j(t) \middle| J_0 = i\right). \tag{11.3}
$$

We know that:

$$
\begin{aligned}
\tilde{R}_{ij}(t) &= \sum_{n=1}^{\infty} P\left(J_n = j, T_n \le t \middle| J_0 = i\right)\\
&= \sum_{n=1}^{\infty} \tilde{Q}_{ik} \bullet Q_{kj}^{(n-1)}(t)\\
&= \sum_{n=1}^{\infty} \tilde{Q}_{ik} \bullet R_{kj}(t).
\end{aligned}
\tag{11.4}
$$

If we now consider the transition probabilities for the delayed semi-Markov process $\left(\tilde{Z}(t), t \ge 0\right)$, associated with the DMRP of triplet $\left(\mathbf{p}, \mathbf{Q}, \tilde{\mathbf{Q}}\right)$,that is:

$$
\tilde{\phi}_{ij}(t) = P\left(\tilde{Z}(t) = j \middle| \tilde{Z}(0) = i\right), \tag{11.5}
$$

using as usual, a simple probabilistic argument, we get:

$$
\tilde{\phi}_{ij}(t) = \delta_{ij}(1 - \tilde{H}_j(t)) + \sum_{k=1}^{m} \int_0^t \phi_{kj}(t-y) d\tilde{Q}_{ik}(y), \tag{11.6}
$$

where of course, the ϕ_{kj} are the transition probabilities of the SMP of kernel $\mathbf{Q}$ and with

$$
\tilde{H}_j(t) = \sum_{k=1}^{m} \tilde{Q}_{jk}(t), j \in I, t \ge 0. \tag{11.7}
$$

Using **Proposition 10.1**, relation (11.4) gives

$$
\tilde{\phi}_{ij}(t) = \delta_{ij}(1 - \tilde{H}_j(t)) + \sum_{k=1}^{m} (1 - H_j) \bullet R_{kj} \bullet \tilde{Q}_{ik}(t), \tag{11.8}
$$

and from relation (11.6):

$$\tilde{\phi}_{ij}(t) = \delta_{ij}(1 - \tilde{H}_j(t)) + (1 - H_j) \bullet \tilde{R}_{ij}(t). \tag{11.9}$$

Relation (11.6) also shows that the limiting distributions of the transition probabilities $\tilde{\phi}_{ij}, i, j \in I$ exist and are known provided the limiting distributions of the transition probabilities $\phi_{ij}, i, j \in I$ exist and are known too.

Indeed, let us suppose that

$$\lim_{t\to\infty} \phi_{ij}(t) = \Pi_{ij},\ i, j \in I, \tag{11.10}$$

then, from relation (11.8), the limits

$$\lim_{t\to\infty} \tilde{\phi}_{ij}(t) = \tilde{\Pi}_{ij}, i, j \in I \tag{11.11}$$

also exist and moreover they are given by

$$\tilde{\Pi}_{ij} = \sum_{k=1}^{m} \tilde{p}_{ik} \Pi_{kj}, i, j \in I, \tag{11.12}$$

where

$$\tilde{p}_{ij} = \tilde{Q}_{ij}(+\infty). \tag{11.13}$$

Using now **Proposition 10.2** and relation (11.11), we get the following result.

Proposition 11.1 *If the MRP associated with the DMRP of triplet* $\left(\mathbf{p}, \mathbf{Q}, \tilde{\mathbf{Q}}\right)$ *is irreducible, then*

$$\lim_{t\to\infty} \tilde{\phi}_{ij}(t) = \Pi_j; i, j \in I \tag{11.14}$$

with

$$\Pi_j = \frac{\pi_j \eta_j}{\sum_k \pi_k \eta_k}, j \in I. \tag{11.15}$$

It follows that in the ergodic case, both DSMP and associated SMP have the same asymptotic behaviour.

As in renewal theory, a very special but very interesting case of the notion of DMRP, is the case of the so-called *stationary MRP* (in short SMRP).This type of process appears when one begins to observe an MRP which has been running a long time so that the first observed r.v. X_1 is in fact the excess γ .

More precisely, let us define the vector $\mathbf{p}_S$ whose jth component (j=1,...,m) is given by

$$p_{S,j} = \Pi_j \tag{11.16}$$

and let us define the semi-Markov kernel $\mathbf{Q}_S$ as follows:

$$Q_{S,ij}(x)=\begin{cases}\frac{1}{\eta_i}\int_0^x (p_{ij}-Q_{ij}(y)dy, & x\geq 0,\\ 0, & x<0.\end{cases} \tag{11.17}$$

We can now give the following definition.

Definition 11.2 *The DMRP of triplet* $\left(\mathbf{p}_S,\mathbf{Q},\tilde{\mathbf{Q}}_S\right)$ *with kernel* **Q** *ergodic, is called a stationary MRP* (*in short SMRP*) *of kernel* **Q**.

In this case, it can be proved that (see Janssen and Manca (2006))

$$P(J_1=k,X_1\leq x)=\frac{1}{\sum_\nu \pi_\nu\eta_\nu}\sum_j \pi_j\int_0^x p_{jk}(1-F_{jk}(y))dy. \tag{11.18}$$

This last result comes from the fact that

$$P(J_1=k,X_1\leq x)=\sum_l p_{S,l}Q_{S,lk}(x) \tag{11.19}$$

and so

$$P(J_1=k,X_1\leq x)=\frac{1}{\sum_\nu \pi_\nu\eta_\nu}\sum_j \pi_j\int_0^x (p_{jk}-Q_{jk}(y))dy, \tag{11.20}$$

which is equivalent to (11.18).

The next propositions give specific properties of SMRP.

Of course, r.v. $N_{S,j}(t)$ represents the total number of passages in state j on $[0,t]$ for the considered SMRP. From this point on, we will systematically use the subscript "S" for parameters related to the SMRP.

Proposition 11.2 *For every SMRP of triplet* $\left(\mathbf{p}_S,\mathbf{Q},\tilde{\mathbf{Q}}_S\right)$ *with kernel* **Q** *ergodic, all renewal functions are linear. More precisely:*

$$E(N_{S,j}(t))=\frac{\Pi_j}{\eta_j}t,\ j\in I. \tag{11.21}$$

Corollary 12.1 *For every SMRP of triplet* $\left(\mathbf{p}_S,\mathbf{Q},\tilde{\mathbf{Q}}_S\right)$ *with kernel* **Q** *ergodic, we have:*

$$E\left(\sum_{j=1}^m N_{S,j}(t)\right)=\frac{t}{\eta}. \tag{11.22}$$

This corollary shows that in looking at the total number of transitions, every SMRP is, on the average, equivalent to a stationary renewal process defined by the d. f. H given by:

$$H = \sum_{j=1}^{m} \pi_j H_j. \tag{11.23}$$

The linearity of Markov renewal functions related to ergodic associated SMRP will give, as the main consequence, the stationarity of the basic stochastic processes related to it.
This means that all marginal distributions will no longer depend on t, and in particular, we have

Proposition 11.3 *For any ergodic SMRP defined by kernel* **Q***, the stochastic process* $\left(Z(t),\gamma(t),t\geq 0\right)$ *is stationary and*

$$P\left(Z(t)=j\right)=\Pi_j. \tag{11.24}$$

12 PARTICULAR CASES OF MRP

We will devote this paragraph to particular cases of MRP having the advantage of leading to some explicit results.

12.1 Renewal Processes And Markov Chains

For the sake of completeness, let us first say that with m=1, that is that the observed system has only one possible state, the kernel **Q** has only one element, say the d.f. **F**, and the process (X_n,n>0) is then a *renewal process*.
Secondly, to obtain *Markov chains* studied in Chapter 2, it suffices to choose for the matrix **F** the following special degenerating case:

$$F_{ij}=U_1,\forall i,j\in I \tag{12.1}$$

and of course an arbitrary Markov matrix **P**.
This means that all r.v. X_n have a.s. the value 1, and so the single random component is the (J_n) process, which is, from relation (3.4) a homogeneous MC of transition matrix **P**.

12.2 MRP Of Zero Order (PYKE (1962))

There are two types of such processes

12.2.1 First Type Of Zero Order MRP

This type is defined by the semi-Markov kernel

$$\mathbf{Q}=\left[p_iF_i\right], \tag{12.2}$$

so that:

$$p_{ij}=p_i,F_{ij}=F_i,j\in I. \tag{12.3}$$

Naturally, we suppose that for every i belonging to I, p_i is strictly positive.
In this present case, we have that the r.v. $J_n, n \geq 0$ are independent and identically distributed and moreover that the conditional interarrival distributions do not depend on the state to be reached, so that, by relation (3.11),

$$H_i = F_i, i \in I. \tag{12.4}$$

Moreover, since:

$$P(X_n \leq x \mid (J_k, X_k), k \leq n-1, J_n) = F_{J_{n-1}}(x), \tag{12.5}$$

we get:

$$P(X_n \leq x \mid (X_k), k \leq n-1) = \sum_{j=1}^{m} p_j F_j(x). \tag{12.6}$$

Introducing the d.f. F defined as

$$F = \sum_{j=1}^{m} p_j F_j, \tag{12.7}$$

the preceding equality shows that, for an MRP of zero order of the first type, the sequence $(X_n, n \geq 1)$ is a renewal process characterized by the d.f. F.

12.2.2 Second Type Of Zero Order MRP

This type is defined by the semi-Markov kernel

$$\mathbf{Q} = \left[p_i F_j \right], \tag{12.8}$$

so that:

$$p_{ij} = p_i, F_{ij} = F_j,\ i, j \in I. \tag{12.9}$$

Here too, we suppose that for every i belonging to I, p_i is strictly positive.
Once again, the r.v. $J_n, n \geq 0$ are independent and equidistributed and moreover the conditional interarrival distributions do not depend on the state to be *left*, so that, by relation (3.11),

$$H_i = \sum_{j=1}^{m} p_j F_j (= F), i \in I. \tag{12.10}$$

Moreover, since:

$$P(X_n \leq x \mid (J_k, X_k), k \leq n-1, J_n) = F_{J_n}(x), \tag{12.11}$$

we get

$$P(X_n \leq x \mid (X_k), k \leq n-1) = \sum_{j=1}^{m} p_j F_j(x) = F(x). \tag{12.12}$$

The preceding equality shows that, for an MRP of zero order of the second type, the sequence $(X_n, n \geq 1)$ is a renewal process characterized by the d.f. F as in the first type;
The basic reason for these similar results is that these two types of MRP are the *reverse*s (timewise) of each other.

12.3 Continuous Markov Processes

These processes are defined by the following particular semi-Markov kernel

$$\mathbf{Q}(x)=\left[p_{ij}\left(1-e^{-\lambda_i x}\right)\right], x\geq 0, \tag{12.13}$$

where $\mathbf{P}=\left[p_{ij}\right]$ is a stochastic matrix and where the parameters $\lambda_i, i\in I$ are strictly positive.

The standard case corresponds to that in which $p_{ii}=0, i\in I$ (see Chung (1960)).

From relation (12.13), we get:

$$F_{ij}(x)=1-e^{-\lambda_i x}. \tag{12.14}$$

Thus the d.f. of sojourn time in state i has an exponential distribution depending uniquely upon the occupied state i, such that both the excess and age processes also have the same distribution.

For m=1, we get the usual Poisson process of parameter λ.

13 A CASE STUDY IN SOCIAL INSURANCE (JANSSEN (1966))

13.1 The Semi-Markov Model

We will now return to the problem presented in section 9.7 of Chapter 2 which was solved using a Markov chain model. Here, we will extend the model to a semi-Markov one allowing us to take into account the duration of passage from invalidity degree S_j to invalidity degree S_k.

In this case, relation (9.114) of Chapter 2 is replaced by the following one:

$$\overline{S}_i'(t)=\sum_{j=1}^{m}\phi_{ij}(t)S_j. \tag{13.1}$$

The study of the financial equilibrium of the funds thus depends on:

$$\overline{S}_i'=\lim_{t\to\infty}S_i'(t), \tag{13.2}$$

or

$$\overline{S}_i'=\sum_{j=1}^{m}\lim_{t\to\infty}\phi_{ij}(t)S_j. \tag{13.3}$$

It suffices here to apply **Proposition 10.3** in order to get the result in the general case.

For the case of uni-reducibility, with **P** represented by relation (9.117) of Chapter 2, we get:

$$\overline{S}_i^{'} = \sum_{j=3}^{5} \Pi_j S_j, \tag{13.4}$$

with

$$\Pi_j = \frac{\pi_j \eta_j}{\sum_{k=3}^{5} \pi_k \eta_k}, \tag{13.5}$$

(π_3, π_4, π_5) being given by relation (9.120) of Chapter 2 and with

$$\eta_j = \sum_{k=3}^{5} p_{jk} \beta_{jk}, \tag{13.6}$$

β_{jk} being the mean of the d.f. F_{jk} related to the duration of passage from invalidity degree S_j to invalidity degree S_k.
Here too, although we have uni-reducibility, the asymptotic state $\overline{S}_i$ is independent of the initial state *i*.
To pass from the Markov chain model to the semi-Markov one, the additional information needed is knowledge of the matrix $\mathbf{B} = \left[\beta_{ij} \right]$.

13.2 Numerical Example

In this section we return to the examples mentioned at the end of Chapter 2 to give the results obtained in the Markov case and also in a semi-Markov environment.
First we have the asymptotic result related to the disability example shown in section 9 of Chapter 2. The means $\eta_i, i \in I$ were computed from real data from Campania, an Italian region with more than 4 million inhabitants, and were applied to the M.C. used for the example.
The semi-Markov asymptotic limit vector is given in **Table 13.1**.

States	η_i	Π_i
1	2.00822	0.00000
2	3.35343	0.00000
3	3.34247	0.22000
4	3.46575	0.34217
5	3.32603	0.43783

Table 13.1: disability Π_i

From result (13.4) and with the degree given in section 9.7 of Chapter 2, we get the following result:

$$\overline{S}_i^{'} = 0.787, \tag{13.7}$$

more or less equivalent to result (9.122) of Chapter 2.

14 (J-X) PROCESSES

In Section 1, we introduced the concept of (J,X) *process* with **Definition 2.4** for which the r.v. X_n, n=1,2,...take their values in the whole real line $\mathbb{R}$ instead of in $\mathbb{R}^+$ for what we called in **Definition 2.2** a *positive* (J,X) *process.*
In fact, in 1969, Janssen showed that the consideration of (J,X) processes leads to a very interesting generalisation of the classical concept of random walk with a lot of applications in stochastic modelling. Let us begin this section by recalling the basic definition.

Definition 14.1 *Let* p=($p_1,...,p_m$) *be an m-dimensional vector of initial probabilities and Q an extended semi-Markov kernel as defined by* **Definition 2.3**. *Then every two-dimensional process* $((J_n, X_n), n = 0,1,...)$ *with values in* $I \times \mathbb{R}$ *and satisfying the conditions*
$P(X_0=0)=1$, a.s.,

$$P(J_0=i)=p_i,\ i=1,...,m \text{ with } \sum_{i=1}^{m} p_i = 1, \tag{14.1}$$

for all $n>0, j=1,...,m$, *we have*:

$$P(J_n = j, X_n \le x \,|\, (J_k, X_k), k = 0,...,n-1) = Q_{J_{n-1}j}(x), a.s., \tag{14.2}$$

is called a (J,X) *process or an extended semi-Markov chain* (in short, ESMC).

From this definition, it follows that we can no longer represent the sample paths of such a process with step functions, as the r.v. X_n can be positive or negative but we can see the S-process defined by:

$$S_n = X_0 + X_1 + \cdots + X_n, n = 0,1,... \tag{14.3}$$

as the successive *positions* of a particle moving on a real line and starting from the origin if the r.v. X_n n=0,1,... represents the successive *steps* of this random movement exactly as the interpretation of a classical *random walk* corresponding to the case of m=1.
This leads to the following definition.

Definition 14.2 *The S-process defined by relation* (14.3) *is called a semi-Markov random walk* (*in short an SMRW*).

It is clear that basic results on positive (J,X) processes given in the preceding chapter are still valid here provided that these properties do not involve the non-negativity of the X_n.
The following proposition summarises the basic properties.

Proposition 14.1 (*Basic properties of* (*J,X*) *processes*)

(i) *The process* $\left((J_n,S_n),n\geq 0\right)$ *is a Markov process with* $I\times\mathbb{R}$ *as state space; more precisely, we have a.s. for all j of I and all real x:*

$$P\left(J_n=j,S_n\leq x\middle|(J_k,S_k),k=0,1,...,n-1\right)=Q_{J_{n-1}j}(x-S_n). \tag{14.4}$$

(ii) *The process* $\left((J_n),n\geq 0\right)$ *is a homogeneous Markov chain with I as state space; more precisely, we have a.s. for all j of I:*

$$P\left(J_n=j,\middle|J_k,k=0,1,...,n-1\right)=p_{J_{n-1}j}. \tag{14.5}$$

(iii) *For all strictly positive n and for all real x, we have:*

$$\begin{aligned}
&P\left(X_n\leq x,\middle|J_k,k=0,1,...,n-1\right)=H_{J_{n-1}}(x),n>0,x\in\mathbb{R},\\
&P\left(X_n\leq x,\middle|J_k,k=0,1,...,n\right)=F_{J_{n-1}J_n}(x),n>0,x\in\mathbb{R},\\
&P\left(X_{n_1}\leq x_1,...,X_{n_l}\leq x_l,\middle|J_k,k=0,1,...,n_k\right)=\prod_{i=1}^{k}F_{J_{n_i-1}J_{n_i}}(x_i),\\
&(0<n_1<\cdots<n_k,x_i\in\mathbb{R},i=1,...,k).
\end{aligned} \tag{14.6}$$

Of course, probabilities p_{ij}, $i,j=1,...,m$, and functions H_j, $j=1,...,m$, F_{ij}, $i,j=1,...,m$ are defined exactly as in relations (2.3) and (3.11).
The last relation in (14.6) shows the conditional independence of $X_{n_1},...,X_{n_k}$ given $J_{n_1-1},J_{n_1},...,J_{n_k}$.
From the relation (3.11) saying that:

$$F_{ij}(x)(=P(X_n\leq x|J_{n-1}=i,J_n=j))=\begin{cases}\dfrac{Q_{ij}(x)}{p_{ij}},p_{ij}>0,\\ U_1(x),p_{ij}=0,\end{cases} \tag{14.7}$$

$$H_i(x)(=P(X_n\leq x|J_{n-1}=i))=\sum_{j=1}^{m}p_{ij}F_{ij}(x),$$

it is clear that a (*J,X*) process is completely defined by either the pair $(\mathbf{p},\mathbf{Q})$ or by the triple $(\mathbf{p},\mathbf{P},\mathbf{F})$ where:

$$\mathbf{p}=(p_1,...p_m),\mathbf{Q}=\left[Q_{ij}\right],\mathbf{P}=\left[p_{ij}\right],\mathbf{F}=\left[F_{ij}\right]. \tag{14.8}$$

15 FUNCTIONALS OF (J-X) PROCESSES

This section introduces the concept of functional *W* of a given (*J,X*) process, fundamental for a lot of applications not only in finance and insurance but also in operations research.
To define the functional *W*, we introduce a real and Lebesgue measurable function *f* of three variables defined on the set $I\times I\times\mathbb{R}$.

When they exist, we will use the following notation for the expectations with $i,k \in I$:

$$\begin{aligned}&\xi_{ik} = \int_{\mathbb{R}} f(i,k,x)dQ_{ik}(x),\ \xi_{ik}^{(2)} = \int_{\mathbb{R}} f^2(i,k,x)dQ_{ik}(x),\\&\xi_i = \sum_{k=1}^{m} \xi_{ik},\ \xi_i^{(2)} = \sum_{k=1}^{m} \xi_{ik}^{(2)}.\end{aligned} \tag{15.1}$$

Definition 15.1 *Given a (J,X) process* $\left((J_n, X_n), n \geq 0\right)$ *defined by* $(\mathbf{p},\mathbf{Q})$, *and a Lebesgue measurable real function f on* $I \times I \times \mathbb{R}$ *, the functional* W_f *is defined as the stochastic process*

$$W_f = (W_f(n), n = 0,1,...) \tag{15.2}$$

where

$$W_f(n) = \begin{cases} 0, n = 0, \\ \sum_{k=1}^{n} f(J_{k-1}, J_k, X_k), n > 0. \end{cases} \tag{15.3}$$

Janssen (1969b) extended to functionals of (*J*,*X*) processes key results obtained by Pyke and Schaufele (1964) for functionals of positive (*J*,*X*) processes: the *strong law of large numbers* (in short SLLN) and the *Central Limit Theorem* (in short CLT).

The basic idea of the proofs is to decompose the sum $W_f(n)$ by introducing *return times* of *return indices* for the embedded Markov chain supposed to be ergodic $\left(J_n, n \geq 0\right)$ defined by relations (9.44), (9.45) and (9.47) in Chapter 2, that is for all *j* belonging to *I*:

$$\begin{aligned}&r_0^{(j)} = 0,\\&r_n^{(j)} = \sup_k \left\{k \in \mathbb{N}_0 : k > r_{n-1}^{(j)}, J_l \neq j, r_{n-1}^{(j)} < l < l\right\}, n \in \mathbb{N}_0.\end{aligned} \tag{15.4}$$

We know that the assumption of ergodicity implies that all states are positive recurrent so that, for all *j*,

$$n \to \infty \Rightarrow r_n^{(j)} \to \infty . \tag{15.5}$$

Moreover, if for all *j* we introduce the stochastic process in discrete time:

$$\left(u_s^{(j)}, n > 0\right) \tag{15.6}$$

with

$$u_s^{(j)} = \sum_{n=r_s^{(j)}+1}^{r_{s+1}^{(j)}} f(J_{n-1}, J_n, X_n), \tag{15.7}$$

then it is clear that this process (15.7) is a sequence of independent and identically distributed r.v. with values in $\mathbb{R}$, that is a *random walk* on the real line.

The following important propositions give some results concerning the moments of r.v. $u_1^j, j \in I, J_0 = j$ and the fundamental *Strong law of large numbers for (J,X) processes.*

Proposition 15.1 *If the embedded Markov chain of the considered (J,X) process is ergodic and if the considered functional is such that the expectations* $\xi_i, \xi_i^{(2)}$ *exist for all i belonging to I, then* $E\left(u_1^{(j)}\right), E\left((u_1^{(j)})^2\right)$ *exist and are given by:*

$$\begin{aligned} E\left(u_1^{(j)}\right) &= \frac{1}{\pi_j}\sum_{i=1}^{m}\pi_i\xi_i (= \mu_{jj}), \\ E\left((u_1^{(j)})^2\right) &= \frac{1}{\pi_j}\sum_{i=1}^{m}\pi_i\xi_i^{(2)} + \frac{2}{\pi_j}\sum_{i=1}^{m}\sum_{k\neq j}\sum_{r\neq j}\pi_i\pi_r(m_{kj}+m_{jr}-m_{kr})\xi_{ik}\xi_r, \end{aligned} \tag{15.8}$$

mean return times in the embedded MC $m_{ls}, l \neq s, l, s \in I$ *being given by relations* (9.42) of Chapter 2.

Proposition 15.2 (*Strong law of large numbers for functionals of (J,X) processes*)
For any ergodic (*J,X*) *process so that the conditional means* $b_{ij}, i, j \in I$ *are finite, we have the following result:*

$$\frac{W_f(n)}{n} \xrightarrow[n\to\infty]{} \sum_{i=1}^{m}\pi_i\xi_i, a.s. \tag{15.9}$$

The next results are related to a central limit theorem for functionals of (*J,X*) processes.

Proposition 15.3 (*Central limit theorem for functionals of (J,X) processes*)
If the expectations ξ_i *exist for all i belonging to I, we have in the ergodic case and for the convergence in law that:*

$$\frac{W_f(n) - n\dfrac{m_j}{m_{jj}}}{\sqrt{n}} \underset{n\to\infty}{\to} N\left(0, \operatorname{var} u_1\left(f - \frac{m_j}{m_{jj}}\right)\right). \tag{15.10}$$

Moreover, if μ_{jj} *defined by the first relation of* (15.8) *is non-null, then*

$$\frac{W_f(n) - nA_f\mu}{\sqrt{n}} \xrightarrow[n\to\infty]{} N(0, \mu B_g) \tag{15.11}$$

where

$$\mu = \sum_{i=1}^{m} \pi_i \eta_i,$$

$$A_f = \frac{\sum_{i=1}^{m} \pi_i \xi_i}{\sum_{i=1}^{m} \pi_i \eta_i}, \tag{15.12}$$

$$B_f = \frac{1}{\sum_i \pi_i \eta_i} \left(\sum_i \pi_i \xi_i^{(2)} + 2 \sum_i \sum_{k \neq j} \sum_{r \neq j} \pi_i \xi_{ik} \xi_r \, {}_j p_{kr}^* \right),$$

$${}_j p_{kr}^* = \frac{m_{kj} + m_{jr} - m_{kr}}{m_{rr}}, i, j, k \in I$$

and $\operatorname{var} u_1 \left(f - \frac{m_j}{m_{jj}} \right)$ *represents the variance of the function* u_1^j *defined by the relation (15.7) but here related to the function* $(f - \frac{m_j}{m_{jj}})$.

Remark 15.1 It can be proved that A_f and B_f are independent of state j.

Proposition 15.4 (*Central limit theorem for the two-dimensional process* $\left(J_n, W_f(n), n \geq 0\right)$)
If the expectations ξ_i *exist for all i belonging to I, we have in the ergodic case and for the convergence in law that:*

$$P\left(J_n = k, \frac{W_f(n) - nA_f \mu}{\sqrt{n}} \leq x \right) = \pi_k \Phi(x), \tag{15.13}$$

where the function Φ *is the distribution function of the normal law* $N(0, \mu B_g)$.

Remark 15.2 An immediate consequence of this last proposition is that processes $(J_n, n \geq 0)$ and $(W_f(n), n \geq 0)$ are asymptotically independent.

These last two propositions immediately give the following one for the special case of (*J*,*X*) processes

Proposition 15.5 (*Central limit theorems for the two-dimensional (J,X) processes*)
For any ergodic (*J*,*X*) *process such that the conditional variances* σ_{ij}^2 *related to the conditional d.f.* $F_{ij}, i, j \in I$ *are finite, we have the following results:*

$$\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow[n\to\infty]{} N(0,\mu B_g), \tag{15.14}$$

$$P\left(J_n = k, \frac{S_n - n\mu}{\sqrt{n}} \le x \right) = \pi_k \Phi(x), \tag{15.15}$$

where the function Φ *is the distribution function related to the normal law* $N\left(0,\sqrt{\mu B_g}\right)$ *with here, for all i and k belonging to I:*

$$\begin{aligned} & g(i,k,x) = x - \sum_{i=1}^{m} \pi_i \eta_i , \\ & \xi_{ik} = p_{ik} b_{ik}, \xi_i = \eta_i , \\ & \xi_{ik}^{(2)} = \left(p_{ik}{}^2 (\sigma_{ik}^2 - b_{ik}{}^2)\right) - 2\left(\sum_{i=1}^{m} \pi_i \eta_i \right) p_{ik} + \left(\sum_{i=1}^{m} \pi_i \eta_i \right)^2 p_{ik}, \end{aligned} \tag{15.16}$$

$\sigma_{ik}^2 (i,k \in I)$ *being the conditional variance related to the conditional distribution* $F_{ik}(i,k \in I)$.

16 FUNCTIONALS OF POSITIVE (J-X) PROCESSES

It is clear that all the results of the preceding paragraph are valid for the special case of positive (J,X) processes for which the r.v. X_n are a.s. non-negative.
But moreover instead of considering the sum of the first n transitions to define $W_f(n)$ in relation (15.3) we can reinsert the *time* with a sum up to $N(t)$, that is the total number of transitions in the semi-Markov process related to the considered semi-Markov kernel **Q**.
In fact , this was the case originally considered by Pyke and Schaufele (1964) so that now, relation (15.3) takes the form:

$$W_f(t) = \begin{cases} 0, N(t) = 0, \\ \sum_{k=1}^{N(t)} f(J_{n-1}, J_n, X_n), N(t) > 0. \end{cases} \tag{16.1}$$

These authors proved the next proposition corresponding to the strong law of large numbers and central limit theorem of the preceding sections.

Proposition 16.1 (*Strong law of large numbers for functionals of positive (J,X) processes*)
If the expectations ξ_i *exist for all i belonging to I, we have in the ergodic case that:*

$$\frac{W_f(t)}{t} \xrightarrow[t\to\infty]{} \frac{m_j}{m_{jj}}, a.s., \tag{16.2}$$

with:

$$m_j = \sum_{i=1}^{m} \pi_i \xi_i,$$
$$m_{jj} = \sum_{i=1}^{m} \pi_i \eta_i, \tag{16.3}$$

the limit ratio m_j / m_{jj} *being still independent of* j.

The propositions related to the central limit theorem have similar extensions for n replaced by $N(t)$ to the numerator and by t to the denominator.

17 CLASSICAL RANDOM WALKS AND RISK THEORY

17.1 Purpose

In the beginning of this chapter, we focused our attention on semi-Markov chains defined by a positive (J,X) process. The case of an extended semi-Markov chain is considered in section 14 starting from a general (J,X) process and having a very different interpretation, directly related to the classical notion of *random walk*.

In the next subsections, we will recall some basic notions concerning random walks that will be extended to the main results of what will be called *Markov random walks* in the next section.

After that, we will develop the main classical models in *risk theory*, which is very useful for insurance companies.

17.2 Basic Notions On Random Walks

Let $\left(X_n, n \geq 1\right)$ be i.i.d. random variables, with F as common d.f., such that:

$$F(0) < 1, \tag{17.1}$$
$$F(0) > 0. \tag{17.2}$$

These two relations imply that for all n, the events

$$\{\omega : X_n > 0\}, \{\omega : X_n < 0\} \tag{17.3}$$

have strictly positive probabilities.

As usual, let us define the following r.v.:

$$S_0 = X_0 = 0, \text{ a.s.}, \tag{17.4}$$

$$S_n = \sum_{k=0}^{n} X_k . \tag{17.5}$$

We can now give the following basic definition:

Definition 17.1 *The random sequence* $\left(S_n, n \geq 0\right)$ *is called a random walk starting at* x_0*, whose* $\left(X_n, n \geq 1\right)$ *are the successive steps.*

If x_0=0, the random walk is said *to start at the origin*.

Example 17.1 If the distribution of r.v. X_n is concentrated on a two-point set $\{-1,1\}$ with

$$p = P(X_n = 1),\ q(=1-p) = P(X_n \neq 1), \tag{17.6}$$

then the associated random walk is called the *simple random walk* or the *Bernoulli random walk.*

The interpretation is quite simple: let us consider for instance a physical particle moving on a straight line starting at the origin.
This particle takes a first unit step to the right with probability p or to the left with probability q and so on.
Clearly, the r.v. S_n will give the position of the particle on the line after the nth step.
Though very particular, the notion of a simple random walk has a lot of important applications in insurance, finance and operations research. A very classical application is the so-called *gambler's ruin problem.*
Let us consider a game with two players such that at each trial, each gambler wins 1 monetary unit with probability p and loses −1 monetary unit with probability $q(=1-p)$.
If u is the initial "fortune" of one player, he will be ruined *at trial n* iff, for the first time, his fortune just after this trial becomes strictly negative.
He will be ruined *before or at trial n* iff he is ruined at one time k, $k \leq n$.
The probability of this last event will be noted by $\Psi(u,n)$ and the probability of being ruined precisely at time n will be noted by $\upsilon(u,n)$.
Clearly, we have:

$$\Psi(u,n) = \sum_{k=0}^{n} \upsilon(u,k) \tag{17.7}$$

and

$$\upsilon(u,n) = \Psi(u,n) - \Psi(u,n-1). \tag{17.8}$$

The probability of not being ruined on $\left[0,n\right]$, that is to say after any trial on $\left[0,n\right]$, will be represented by $\gamma(u,n)$, and of course, we have:

$$\gamma(u,n) = 1 - \Psi(u,n). \tag{17.9}$$

Probabilities $\gamma(u,n)$ and $\Psi(u,n)$ are called respectively the *non-ruin probability* and the *ruin probability* on $\left[0,n\right]$ starting at time 0 with an initial fortune - also called *reserve* or *equities* for insurance companies - of amount u.

Now we will see how to express these two probabilities with the aid of events as functions of the variables X_n, n=0,1,.. representing the "gain" (positive or negative) of the considered player just after the nth trial.
Starting with $x_0=u$, we can write:

$$\upsilon(u,n) = P\left(S_k > 0, k = 1,...,n-1, S_n < 0\right). \tag{17.10}$$

If we introduce now the discrete r.v. T defined as follows:

$$T = \inf\left\{n : S_n < 0\right\}, \tag{17.11}$$

we get:

$$\upsilon(u,n) = P\left(T = n\right), \tag{17.12}$$

$$\Psi(u,n) = P\left(T \leq n\right), \tag{17.13}$$

$$\gamma(u,n) = P\left(T > n\right). \tag{17.14}$$

One of the major problems in the so-called *risk theory* is that of giving explicit results about these three probabilities. This part of risk theory is called the *ruin problem* and it is equivalent to studying the distribution of the stopping time T.
It is recalled in Janssen and Manca (2006) that the main concepts in random walk theory are those of *ladder variables*. To be self-contained, let us briefly give the following basic **Definition 17.2**.
Graphically, *ladder epochs* and *ladder heights* are clearly seen on trajectories given in a two-dimensional graph where points are designated by the coordinates $\left\{(n, S_n), n \geq 0\right\}$.

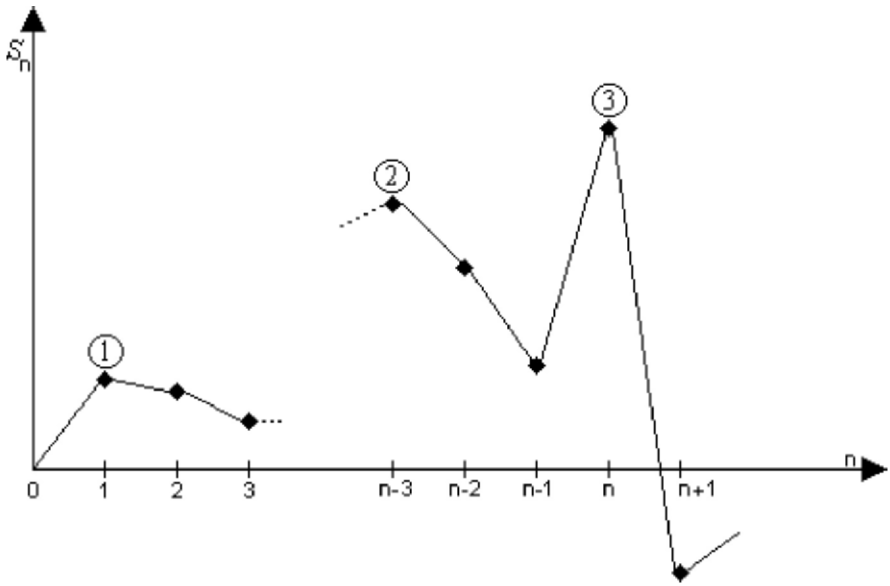

Figure 17.1: ladder trajectory

In the graph of **Figure 17.1**, we have joined the points of coordinates $(k,S_k),(k+1,S_{k+1})$ in order to clearly show the evolution of the process.
For example, in the trajectory of **Figure 17.1**, we have the strictly ascending ladder points $(1,S_1),(n-3,S_{n-3}),(n,S_n)$ and the strictly descending ladder points $(n-1,S_{n-1}),(n+1,S_{n+1})$.
The next definitions formalise these concepts of ladder variables. We follow the presentation of Feller (1971) but without the assumption that x_0=0.

Definition 17.2 *The first (strict) ascending ladder point* (Γ_1,H_1) *is the first term of the sequence* $((n,S_n),n>0)$ *for which* S_n *is strictly superior to* S_0*. That is:*

$$\{\omega:\Gamma_1=n\}=\{\omega:S_1\le S_0,...,S_{n-1}\le S_0,S_n>S_0\}. \tag{17.15}$$

The r.v. Γ_1 *is called the first strict ladder epoch and the r.v. defined by*

$$H_1=S_{\xi_1},\ \xi_1=\Gamma_1 \tag{17.16}$$

is called the first strict ladder height.

The possibly defective bi-dimensional distribution of (Γ_1,H_1) will be noted by

$$H_n(x)=P(\Gamma_1=n,H_1\le x),n>0,x\in\mathbb{R}^+\cup\{+\infty\}. \tag{17.17}$$

Consequently, we obtain:

$$P(\Gamma_1=n)=H_n(+\infty), \tag{17.18}$$

$$P(H_1\le x)=\sum_{n=1}^{\infty}H_n(x)(=M(x)), \tag{17.19}$$

so that both r.v. Γ_1,H_1 have the same defect; that is:

$$P(\Gamma_1=\infty)=P(H_1=\infty)=1-M(\infty). \tag{17.20}$$

17.3 Classification Of Random Walks

This section is devoted to a very important result known as the *classification of random walks*. Briefly, this result states that only two possibilities exist for the asymptotic behaviour of the random walk $(S_n,n\ge 0)$.
Either:

$$P(\limsup S_n=\infty)=P(\liminf S_n=-\infty)=1 \tag{17.21}$$

or:

$$P(\lim S_n=\infty \text{ or } \lim S_n=-\infty)=1. \tag{17.22}$$

In the first case, the random walk is called *oscillating*; in the second, it is said to *drift to* $+\infty$ or $-\infty$.
In this last possibility, we have a.s.

$$\lim S_n = +\infty \tag{17.23}$$

or

$$\lim S_n = -\infty \ . \tag{17.24}$$

Proposition 17.1 *There exist only two types of random walks:*
(1) *the oscillating type: both ascending and descending renewal processes of ladder heights are persistent. In this case, the process* $(S_n,\, n = 0,1,\ldots)$ *oscillates with probability* 1 *between* $-\infty$ *and* $+\infty$, *and:*

$$E\left(\Gamma_1\right) = E\left(\Gamma_1^D\right) = \infty\,; \tag{17.25}$$

(2) *drift toward* $\pm\infty$: *in the case of* $-\infty$, *the ascending renewal process is terminated and the descending renewal process is proper with probability* 1. *The process* $(S_n,\, n = 0,1,\ldots)$ *drifts with probability* 1 *toward* $-\infty$ *and reaches a finite non-negative maximum. M; moreover:*

$$E\left(\Gamma_1^D\right) = \frac{1}{1-\zeta}\varsigma(\infty) = \frac{1}{1-\zeta}\frac{1}{1-M(\infty)}. \tag{17.26}$$

And if M is the r.v. defined as

$$M = \max(S_0, S_1, ..., S_n, ...), \tag{17.27}$$

then

$$P(M \le x) = (1 - M(\infty))\varsigma(x)\,. \tag{17.28}$$

The results are analogous in the case of a drift toward $+\infty$.

In fact, when the mean μ of the r.v. X_n, $n>1$ exists, the strong law of large number asserts that, a.s.:

$$\lim_{n\to\infty}\frac{S_n}{n} = \mu, \tag{17.29}$$

so that we immediately get the following results:

$$\begin{aligned} \mu > 0 &\Rightarrow \lim_{n\to\infty} S_n = +\infty, \\ \mu < 0 &\Rightarrow \lim_{n\to\infty} S_n = -\infty. \end{aligned} \tag{17.30}$$

The next theorem gives the complete relation, including the case $\mu = 0$ which is more difficult to treat.

Proposition17.2 If *the mean* μ *of the r.v.* X_n*, n>1 exists, then:*
(i) μ *=0 implies that the random walk is oscillating,*
(ii) μ >0 *implies that the random walk drifts toward* $+\infty$,
(iii) μ <0 *implies that the random walk drifts toward* $-\infty$.

In fact, it can also be proved that the converse is true, and so we have:

(i) $\mu=0 \Leftrightarrow P\left(\limsup_{n} S_n = +\infty, \liminf_{n} S_n = -\infty\right) = 1$, (17.31)

(ii) $\mu>0 \Leftrightarrow P\left(\lim_{n\to\infty} S_n = +\infty\right) = 1$, (17.32)

(iii) $\mu>0 \Leftrightarrow P\left(\lim_{n\to\infty} S_n = -\infty\right) = 1$. (17.33)

18 DEFECTIVE POSITIVE (J-X) PROCESSES

To extend the main results of classical random walks on semi-Markov random walks, it is necessary to introduce now the concepts of defective and terminated positive (*J*,*X*) processes.

In fact we will now consider a new type of two-dimensional process always defined by a kernel **Q** but for which the matrix **P** of the embedded Markov chain $(J_n, n \geq 0)$ could be *substochastic*, i.e. such that at least the sum of the elements of at least one line is strictly inferior to 1.

To do this, we formally introduce a supplementary state to the set *I* called state 0 which is *absorbing* so that instead of *I*={1,…,*m*} as state space, we work now with the new state space

$$\overline{I} = I \cup \{0\} \tag{18.1}$$

and with a new matrix $(m+1)\times(m+1)$, represented by $\dot{\mathbf{P}}$ defined from the given substochastic matrix **P** as follows:

$$\begin{aligned} &\dot{p}_{ij} = p_{ij}, i, j \in I, \\ &\dot{p}_{ok} = \delta_{ok}, k \in \overline{I}, \\ &\dot{p}_{j0} = 1 - \sum_{k=1}^{m} p_{jk} (= 1 - \upsilon_j). \end{aligned} \tag{18.2}$$

For the sojourn times, when the embedded MC reaches the absorbing state 0, we define the sojourn time as being infinite and so the associated *T* process terminates.

This leads to the following definition.

Definition 18.1 *A defective positive* (*J,X*) *process* $((J_n, X_n), n \geq 0)$ *with state space* $\overline{I}\times\mathbb{R}^+$ *is defined by a vector of initial distributions* $\dot{\mathbf{p}} = (p_0, p_1, ..., p_m)$ *and a kernel* $(m+1)\times(m+1)$ $\ddot{\mathbf{Q}} = \left[\ddot{Q}_{ij}\right]$ *of mass functions on the positive half real line such that the* $m \times m$ *sub-matrix*

$$\ddot{\mathbf{Q}} = \begin{bmatrix} \ddot{Q}_{11} & \cdots & \ddot{Q}_{mm} \\ \cdots & \cdots & \cdots \\ \ddot{Q}_{m1} & \cdots & \ddot{Q}_{mm} \end{bmatrix} \tag{18.3}$$

is a positive semi-Markov kernel and

$$\ddot{Q}_{i0}(x)=\begin{cases}0, & x\in\mathbb{R},\\ \dot{p}_{i0}, & x=+\infty,\end{cases}$$

$$\ddot{Q}_{00}(x)=\begin{cases}0, & x\in\mathbb{R},\\ 1, & x=+\infty,\end{cases}\tag{18.4}$$

$$\ddot{Q}_{0i}(x)=0,\ \ i\in I, x\in\overline{\mathbb{R}}.$$

Moreover, if

$$\ddot{p}_{ij}=\lim_{x\to\infty}\ddot{Q}_{ij}(x),\tag{18.5}$$

the matrix

$$\ddot{\mathbf{P}}=\left[\ddot{p}_{ij}\right]\tag{18.6}$$

is a sub-stochastic matrix satisfying conditions (18.2).

The defective positive (J,X) process satisfies the conditions

$$P(X_0=0)=1, \text{ a.s.},\tag{18.7}$$

$$P(J_0=i)=p_i\ \ i=1,\ldots,m \text{ with } \sum_{i=1}^{m}p_i=1,\tag{18.8}$$

and for all $n>0, j=1,\ldots,m,$ we have:

$$P(J_n=j, X_n\le x\,|\,(J_k,X_k),k=0,\ldots,n-1)=\ddot{Q}_{J_{n-1}j}(x),\ a.s.\tag{18.9}$$

Let us recall that when the process enters for the first time state 0, it will stay forever and so we say that the process *terminates* or *is terminated* at the first entrance time in state 0; so, we can define the *unconditional waiting time distributions* as follows:

$$\ddot{H}_i(x)=P(X_n\le x\,|\,J_{n-1}=i)=\sum_{j\in I}\ddot{Q}_{ij}(x), i\in I,$$

$$\ddot{H}_0(x)=P(X_n\le x\,|\,J_{n-1}=0)=\begin{cases}0, & x\in\mathbb{R},\\ 1, & x=+\infty,\end{cases}\tag{18.10}$$

and consequently:

$$P(X_n=+\infty\,|\,J_{n-1}=i)=P(J_n=0\,|\,J_{n-1}=i)=1-\upsilon_i, i\in I,\tag{18.11}$$

and of course:

$$\lim_{x\to\infty}\ddot{H}_i(x)=\upsilon_i, i\in I.\tag{18.12}$$

Now, let $T_{ij}(x,n)(i,j\in I)$ be the probability that the (J,X) process terminates at step n with $J_n=j$, before or at time x given that $J_0=i$.
It is clear that:

$$T_{ij}(x;n)=P(T_n\le x, J_n=j, J_{n+1}=0\,|\,J_0=i),$$
$$=(1\text{-}\upsilon_j)(Q_{ij}^{(n)}(x)).\tag{18.13}$$

The value of the probability $T_{ij}(n)(i,j \in I)$ that the (J,X) process terminates at step n with $J_n=j$, given that $J_0=i$, is

$$T_{ij}(n) = \lim_{x\to\infty} T_{ij}(x;n) = p_{ij}^{(n)}(1-\upsilon_j), \tag{18.14}$$

the value of the probability $T_i(n)(i \in I)$ that the (J,X) process terminates at step n given that $J_0=i$, is

$$T_i(n) = \sum_{j\in I} T_{ij}(n) = \sum_{j\in I} p_{ij}^{(n)}(1-\upsilon_j), \tag{18.15}$$

and finally the probability $T_i, (i \in I)$, that the (J,X) process terminates given that $J_0=i$, is given by

$$T_i = \sum_{n=0}^{\infty} T_i(n) = \sum_{n=0}^{\infty}\sum_{j\in I} p_{ij}^{(n)}(1-\upsilon_j). \tag{18.16}$$

Replacing the $\upsilon_j, j \in I$ by (18.2), we find the result that:

$$\begin{aligned} &T_i(n) = \dot{p}_{i0}^{(n+1)}, \\ &T_i = \lim_{n\to\infty} \dot{p}_{i0}^{(n+1)}. \end{aligned} \tag{18.17}$$

Using a result of Chung (1960, p.227), we have that the $(T_i, i \in I)$ satisfy the following linear system:

$$T_i = \sum_{k\in T} \dot{p}_{ik} T_k + \dot{p}_{i0}, i \in T, \tag{18.18}$$

where T is the set of transient states in the imbedded Markov chain.

Definition 18.2

(i) *A defective positive* (J,X) *process is i-terminated iff, starting from state i, it terminates a.s.*

(ii) *A defective positive* (J,X) *process defined by the triple* $(m,\mathbf{p},\mathbf{Q})$ *is p-terminated iff, starting from initial distribution* **p**, *it terminates a.s.*

(iii) *A class of defective positive* (J,X) *process defined by the doublet* $(m,\mathbf{Q})$ *is terminated iff, starting with any initial distribution* **p**, *it is p-terminated.*

The next proposition is now obvious.

Proposition 18.1

(i) *A defective positive* (J,X) *process is i-terminated iff* $T_i=1$.

(ii) *The defective positive* (J,X) *process* $(m,\mathbf{p},\mathbf{Q})$ *is p-terminated iff* $\sum_{i\in I} p_i T_i = 1$.

(iii) *The class defective positive* (J,X) *process is terminated iff* $T_i=1$, *for all i belonging to I.*

Remark 18.1 If state 0 is the only recurrent state, then $T_i=1$ for all i of I and so we have a terminal class.

In particular, this is the case when for all i of I, $\upsilon_i < 1$.
For defective positive (J,X) processes, we can introduce the concept of lifetime of the process defined by the r.v. M as follows.

Definition 18.3 *The lifetime of a defective positive (J,X) process is the r.v. M defined as*:

$$M = \sup_{0 \le n \le \sup\{k: J_{k-1} \ne 0\}} T_n . \tag{18.19}$$

It is clear that the conditional distributions of M, having $\overline{\mathbb{R}}$ as support, are given by:

$$M_i(x) = P(M \le x \mid J_0 = i) = \sum_{j \in I} \sum_{n=0}^{\infty} T_{ij}(x;n) . \tag{18.20}$$

Using relation (18.13), we get:

$$\begin{aligned} P(M \le x \mid J_0 = i) &= \sum_{j \in I} (1 - \upsilon_j)(W_{ij}(x)), \\ W(x) &= \sum_{n=0}^{\infty} Q^{(n)}(x), \end{aligned} \tag{18.21}$$

a result giving the theoretical *explicit form* of the lifetime conditional distributions.
A simple probabilistic reasoning proves that the functions M_i, $i=1,\dots,m$ satisfy the following integral equation system of renewal type:

$$M_i(x) = (1 - \upsilon_i) + \sum_{j \in I} \int_0^x M_j(x-y) dF_{ij}(y), i \in I , \tag{18.22}$$

whose (18.21) is the unique solution.
In the case of all the d.f. M_i, being proper (i.e. $\lim_{x \to \infty} M_i(x) = 1, i = 1,\dots,m$) and if, moreover, the following mean lifetimes exist:

$$\bar{M}_i = \int_0^{\infty} x dM_i(x) < \infty, i = 1,\dots,m, \tag{18.23}$$

then, from relation (18.22), it follows that they satisfy the system:

$$\bar{M}_i = \sum_{j=1}^{m} p_{ij} \bar{M}_j + \eta_j, i = 1,\dots,m. \tag{18.24}$$

Example 18.1 Let us consider the class of defective positive (J,X) processes for which the matrix $\mathbf{P} = \left[p_{ij} \right]$ satisfies:

$$\begin{aligned} &p_{ij} = p_j > 0, i, j \in I, \\ &\sum_{j=1}^{m} p_j = a < 1. \end{aligned} \tag{18.25}$$

In this case, state 0 is the only recurrent state for the Markov matrix $\dot{\mathbf{P}}$ and so, the considered class of defective positive (*J,X*) processes is terminated with:

$$\begin{aligned} &\upsilon_j = a, \\ &\dot{p}_{i0}^{(n)} = 1 - a^n. \end{aligned} \tag{18.26}$$

It follows then that all the d.f. M_i are proper and, from (18.21) given by

$$M_i(x) = (1-a)\sum_{j=1}^{m} (K(x))_{ij}, \tag{18.27}$$

a result giving a theorem of Feller (1971).
Here, the linear system (18.24) has the following explicit form:

$$M_i = \frac{\begin{vmatrix} (p_1 - \delta_{i1}) & & & p_m & -\eta_1 \\ & (p_2 - \delta_{i1}) & & & -\eta_2 \\ & & (p_i - 1) & & \\ p_1 & & & (p_m - \delta_{i1}) & -\eta_m \end{vmatrix}}{(-1)^m (1 - \sum_{j=1}^{m} p_j)}, i = 1, ..., m. \tag{18.28}$$

If moreover, the considered class here is of *order* 0, this means we have as supplementary condition on sojourn conditional distributions:

$$F_{ij} = F_j, i, j = 1, ..., m, \tag{18.29}$$

then from (18.28) the M_i functions do not depend on i and the common value $\bar{M}$ of the mean lifetimes $\bar{M}_i$ is given by

$$\bar{M} = \frac{\eta}{1 - \sum_{j=1}^{m} p_j}, \tag{18.30}$$

where

$$\eta = \sum_{j=1}^{m} p_j b_j, b_j = \int_0^\infty t dF_j(t), j = 1, ..., m. \tag{18.31}$$

The special case m=1 is already given by Feller (1966).

19. SEMI-MARKOV RANDOM WALKS

Let us consider a (*J,X*) process $((J_n, X_n), n \geq 0)$.

It can be seen that the process (X_n) defines a random walk on the real line starting at X_0=0, a.s. but contrary to the classical random walks, the successive steps X_n are no longer independent but satisfy a semi-Markovian dependence, useful for a lot of applications for example in risk theory, queues and so on.
The position of the particle at the *n*th step is given by the r.v.

$$S_n = \sum_{k=1}^{n} X_k, n = 1,2,\ldots . \tag{19.1}$$

From now on, we will suppose that the imbedded Markov chain $(J_n, n \geq 0)$ is irreducible and so recurrent.
For j belonging to I fixed, let us introduce once more the process $(r_n^{(j)}, n \geq 0)$ called the process of *recurrence indices* as follows:

$$\begin{aligned} &r_0^{(j)} = 0, \\ &r_n^{(j)} = \sup_k \{k \geq 0 : k > r_{n-1}^{(j)}, J_l \neq j, r_{n-1}^{(j)} < l < k\}, n > 0. \end{aligned} \tag{19.2}$$

From section 14, we know that the mean recurrence times given by :

$$m_{jj} = E\left(r_{n+1}^{(j)} - r_n^{(j)}\right), n > 0, j \in I \tag{19.3}$$

and are here finite.
With the renewal process $(r_{n+1}^{(j)} - r_n^{(j)}, n > 0)$, we can associate a classical random walk, i.e. a sequence of i.i.d. random variables:

$$(U_l^{(j)}, l > 0) \tag{19.4}$$

with

$$U_l^{(j)} = \sum_{n=r_l^{(j)}+1}^{r_{l+1}^{(j)}} X_n. \tag{19.5}$$

From **Proposition 15.1** and the **Remark 15.1**, it follows that these random variables have a mean μ given by

$$\mu_{jj} = \frac{1}{\pi_j} \sum_{i=1}^{m} \pi_i \eta_i, \tag{19.6}$$

so that they are positive or negative for all j, depending on the sign of μ defined by

$$\mu = \sum_{i+1}^{m} \pi_i \eta_i \ . \tag{19.7}$$

As we already know, following Spitzer (1957) and Feller (1971) we will say that a random walk *drifts to* $+\infty$ *(resp.* $-\infty$*)* iff

$$\begin{aligned} &P(\limsup_n \{\omega : S_n(\omega) < 0\}) = 0, \\ &(P(\liminf_n \{\omega : S_n(\omega) > 0\}) = 0) \end{aligned} \tag{19.8}$$

and that it is *oscillating* iff

$$P(\limsup_n \{\omega : S_n(\omega) < 0\}) = P(\liminf_n \{\omega : S_n(\omega) > 0\}) = 1. \qquad (19.9)$$

We then get the following theorem concerning the asymptotic behaviour of the semi-Markov random walk (S_n).

Proposition 19.1 *If the semi-Markov random walk* (S_n) *has an irreducible M.C. and all the unconditional means* $\eta_i, i \in I$ *are finite and, then if* μ *is null and if for one j,*

$$P(U_1^{(j)} = 0) < 1, \qquad (19.10)$$

then the semi-Markov random walk is oscillating and if μ *is positive (resp. negative) then the semi-Markov random walk drifts to* $+\infty$ *(resp.* $-\infty$*).*

20 DISTRIBUTION OF THE SUPREMUM FOR SEMI-MARKOV RANDOM WALKS

Let us consider a semi-Markov random walk (S_n) with an irreducible M.C. and all the unconditional means $\eta_i, i \in I$ finite. We are now interested in the distribution of the following supremum:

$$M = \sup\{S_0, S_1, ...\}. \qquad (20.1)$$

For $\mu > 0$, under the assumptions of **Proposition 19.1**, it follows from this proposition that for all *i* of *I* and all real *x*:

$$P(M \le x | J_0 = i) = 0. \qquad (20.2)$$

This is also true for μ =0, as the positive (*J,X*) process $((H_n, \varsigma_n), n > 0)$ is regular (see Pyke (1961a)) meaning that it has only a finite number of transitions on any time interval.

Now for μ <0, from relation (5.21), we get:

$$M_i(x) = P(M \le x | J_0 = i) = \sum_j (1 - \upsilon_j) \tilde{M}_{ij}(x), \qquad (20.3)$$

where $\tilde{\mathbf{M}} = \left[\tilde{M}_{ij} \right]$ is the matrix of renewal functions for the process $((H_n, \varsigma_n), n > 0)$.

From **Proposition 7.2** of Chapter 5 of Janssen and Manca (2006), we know that:

$$\lim_{x \to \infty} M_i(x) = 1, \forall i \in I. \qquad (20.4)$$

We also see that

$$M_i(0) = 1 - \upsilon_i, \forall i \in I. \qquad (20.5)$$

Nevertheless, to be useful, the "explicit" form (8.3) requires us to know the kernel of the positive (*J,X*) process $((H_n, \varsigma_n), n > 0)$ or the functions H^{ij} given by relation (7.8). Unfortunately, this is very difficult except in very particular cases.

To avoid that, we can start from the following integral equations system of Wiener-Hopf type given from an immediate probabilistic reasoning:

$$M_i(x) = \begin{cases} \sum_j \int_{-\infty}^{x} M_j(x-s) dQ_{ij}(x), & x \geq 0, \\ 0, & x < 0. \end{cases} \quad (20.6)$$

For m=1, we get the classical Wiener-Hopf equation:

$$M(x) = \begin{cases} \int_{-\infty}^{x} M(x-s) dQ(x), & x \geq 0, \\ 0, & x < 0. \end{cases} \quad (20.7)$$

Janssen (1970) proved that this integral equations system of Wiener-Hopf type has a unique P-solution, meaning a vector $(M_1,...,M_n)$ of distribution functions satisfying system (20.6).

21 NON-HOMOGENEOUS MARKOV AND SEMI-MARKOV PROCESSES

To finish this chapter, let us recall the basic definitions and results for the *non-homogeneous* case for which time itself has influence on the transition probabilities. Due to the importance of applications, in particular in insurance, we carefully develop some special cases such as non-homogeneous Markov processes.

21.1 GENERAL DEFINITIONS

To begin with, we present the general definition of non-homogeneous semi-Markov processes (in short NHSMP) including as particular cases, non-homogeneous Markov processes (in short NHMP) in continuous time, non-homogeneous Markov chains (in short NHMC) in discrete time and non-homogeneous renewal processes (in short NHRP).
We follow the original presentation given by Janssen and De Dominicis (1984).

21.1.1 Completely Non-Homogeneous Semi-Markov Processes

As usual, let us consider a system S having m possible states constituting the set I={1,...,m} defined on the probability space $(\Omega, \Im, P)$.

Definition 21.1 *The two-dimensional process in discrete time* $\left((J_n, X_n), n \geq 0\right)$ *with values in* $I \times \mathbb{R}^+$ *such that:*

$$\begin{aligned} &J_0 = i, X_0 = 0, a.s., i \in I, \\ &P(J_n = j, X_n \leq x \,|\, (J_k, X_k), k \leq n-1) = {}^{(n-1)}Q_{J_{n-1}j}(T_{n-1}, T_{n-1} + x), \\ &j \in I, x \in \mathbb{R}^+, \\ &T_0 = 0, T_n = \sum_{k=0}^{n} X_k, a.s. \end{aligned} \tag{21.1}$$

is called a completely non-homogeneous semi-Markov chain (*in short CNHSMC*) *of kernel* $\mathbf{Q}(s,t) = \left({}^{(n-1)}\mathbf{Q}(s,t), n \geq 1\right)$.

Consequently, the past influences the evolution of the process by the presence of T_{n-1} and n in (21.1).

Definition 21.2 *The sequence* $\mathbf{Q} = \left({}^{(n-1)}\mathbf{Q}(s,t), n \geq 1\right)$ *of* $m \times m$ *matrices of measurable functions of* $\mathbb{N}_0 \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto [0,1]$ *where*:

$${}^{(n-1)}\mathbf{Q}(s,t) = \left[{}^{(n-1)}Q_{ij}(s,t)\right] \tag{21.2}$$

and satisfies the following conditions:

$$\begin{aligned} &\text{(i) } \forall n > 0, \forall i, j \in I, \forall t, s \in \mathbb{R}^+ : t \leq s \Rightarrow {}^{(n-1)}Q_{ij}(s,t) = 0, \\ &\text{(ii) } \forall n > 0, \forall i \in I, \forall s \in \mathbb{R}^+ : \sum_{j=1}^{n} {}^{(n-1)}Q_{ij}(s,\infty) = 1, \\ &\text{with } {}^{(n-1)}Q_{ij}(s,\infty) = \lim_{t \to \infty} {}^{(n-1)}Q_{ij}(s,t), \end{aligned} \tag{21.3}$$

is called a completely non-homogeneous semi-Markov (*in short CNHSM*) *kernel.*

Clearly, for all fixed s, ${}^{(n-1)}Q_{ij}(s,.)$ is a mass function, null for $t \leq s$.

Definition 21.3 *For all* $i, j \in I, n \in \mathbb{N}_0, s, t \in \mathbb{R}^+$, *the functions* ${}^{(n-1)}p_{ij}(s)$, ${}^{(n-1)}H_{ij}(s,t)$, ${}^{(n-1)}F_{ij}(s,t)$ *are defined as follows:*

$$\begin{aligned} &{}^{(n-1)}p_{ij}(s) = {}^{(n-1)}Q_{ij}(s,\infty), \\ &{}^{(n-1)}H_{ij}(s,t) = \sum_{j} {}^{(n-1)}Q_{ij}(s,t), \\ &{}^{(n-1)}F_{ij}(s,t) = \begin{cases} U_1(s)U_1(t), & {}^{(n-1)}p_{ij}(s) = 0, \\ \dfrac{{}^{(n-1)}Q_{ij}(s,t)}{{}^{(n-1)}p_{ij}(s)}, & {}^{(n-1)}p_{ij}(s) > 0. \end{cases} \end{aligned} \tag{21.4}$$

Working as in section 3, it is easy to prove that we still have the following probabilistic meaning:

$$\begin{aligned}
&{}^{(n-1)}p_{ij}(s) = P(J_n = j \,|\, J_{n-1} = i, T_{n-1} = s),\\
&{}^{(n-1)}H_i(s,t) = P(X_n \le t - s \,|\, J_{n-1} = i, T_{n-1} = s)(= P(T_n \le t \,|\, J_{n-1} = i, T_{n-1} = s)),\\
&{}^{(n-1)}F_{ij}(s,t) = P(X_n \le t - s \,|\, J_{n-1} = i, J_n = j, T_{n-1} = s)\\
&\qquad (= P(T_n \le t \,|\, J_{n-1} = i, J_n = j, T_{n-1} = s)).
\end{aligned} \tag{21.5}$$

In matrix notation, using the element by element product (Scott product) defined as:

$$\begin{aligned}
&\mathbf{A} \cdot \mathbf{B} = \left[a_{ij} b_{ij} \right],\\
&\mathbf{A} = \left[a_{ij} \right], \mathbf{B} = \left[b_{ij} \right],
\end{aligned} \tag{21.6}$$

we will write:

$$\begin{aligned}
&{}^{(n-1)}\mathbf{F}(s,t) = \left[{}^{(n-1)}F_{ij}(s,t) \right],\\
&{}^{(n-1)}\mathbf{P}(s) = \left[{}^{(n-1)}p_{ij}(s) \right],\\
&{}^{(n-1)}\mathbf{Q}(s,t) = {}^{(n-1)}\mathbf{P}(s) \cdot {}^{(n-1)}\mathbf{F}(s,t).
\end{aligned} \tag{21.7}$$

We can now give the following definitions similar to the classical or homogenous semi-Markov theory presented in section 5.

Definition 21.4 *The counting process* $(N(t), t \ge 0)$ *defined as*

$$N(t) = \sup_n \left\{ n : T_n \le t \right\} \tag{21.8}$$

is called the associated counting process with the CNHSM kernel **Q**.

Definition 21.5 *The process* $\left((J_n, T_n), n \ge 0 \right)$ *is called a completely non-homogeneous Markov additive process or Markov renewal process* (*in short CNHMAP or CNHMRP*).

Definition 21.6 *The process* $Z = (Z(t), t \ge 0)$ *defined as*

$$Z(t) = \begin{cases} J_{N(t)}, N(t) < \infty, \\ \theta, N(t) < \infty, \end{cases} \tag{21.9}$$

where θ *is a new state added to I, is called the completely non-homogeneous semi-Markov process* (*in short CNHSMP*) *of kernel* **Q**.

Definition 21.7 *The random variable L defined as*

$$L = \inf \left\{ t : Z(t) = \theta \right\} \tag{21.10}$$

is called the lifetime of the CNHSMP Z.

Definition 21.8 *The associated counting process* $(N(t), t \geq 0)$ *or the CNHSMP* $Z = (Z(t), t \geq 0)$ *of kernel* **Q** *is explosive iff*

$$L = \infty, a.s. \tag{21.11}$$

and non-explosive iff

$$L < \infty, a.s. \tag{21.12}$$

For very general counting processes, De Vylder and Haezendonck (1980) have given necessary and sufficient conditions for non-explosion. Here, in general, we always assume non-explosive processes.

For the two-dimensional process $((J_n, T_n), n \geq 0)$, we have the following result:

$$\begin{aligned} &P\left(J_1 = j, T_1 \leq t \middle| J_0 = i, T_0 = 0\right) = {}^{(0)}Q_{ij}(0,t) (= Q_{ij}^{(1)}(t)), \\ &P\left(J_2 = j, T_2 \leq t \middle| J_0 = i, T_0 = 0\right) = \sum_k \int_0^t {}^{(1)}Q_{kj}(x,t)\, {}^{(0)}Q_{ij}(0,dx) (= Q_{ij}^{(2)}(t)), \end{aligned} \tag{21.13}$$

and in general

$$\begin{aligned} P\left(J_n = j, T_n \leq t \middle| J_0 = i, T_0 = 0\right) &= \sum_k \int_0^t {}^{(n-1)}Q_{kj}(x,t)\, {}^{(1)}Q_{ij}(dx) \\ &= (Q_{ij}^{(n)}(t)), n > 1). \end{aligned} \tag{21.14}$$

Using matrix notation, we may write for two $m \times m$ matrices of mass functions **A**(t), **B**(t):

$$\int_0^t \mathbf{A}(t) d\mathbf{B}(t) = \left[\sum_{k=1}^n \int_0^t B_{kj}(z) dA_{ik}(z) \right], \tag{21.15}$$

and so relations (21.14) can be written under the matrix form:

$$\begin{aligned} &\mathbf{Q}^{(n)}(t) = \int_0^t \mathbf{Q}^{(n-1)}(z,t) d\mathbf{Q}^{(1)}(x), n > 1, \\ &\text{with} \\ &\mathbf{Q}^{(n)}(t) = \left[Q_{ij}^{(n)}(t) \right], n \geq 1, \\ &\mathbf{Q}^{(1)}(t) = \left[{}^{(0)}Q_{ij}(0,t) \right]. \end{aligned} \tag{21.16}$$

In the particular class of classical SMP, the relation (21.16) gives the n-fold convolution of the SM kernel **Q**.

Another very important distribution is the marginal distribution of the Z process as it gives the state occupied by the system S at time t.

Let us introduce the following probabilities:

$${}^{(n)}\phi_{ij}(s,t) = P\left(Z(t) = j \middle| Z(0) = i, N(s-) < N(s), N(s) = n\right), i, j \in I, n \geq 0. \tag{21.17}$$

The conditioning means that $T_n = s$ and that there exists a transition at time s such that the new state occupied after the transition is i.

Clearly, these probabilities satisfy the following relations:

$$^{(n)}\phi_{ij}(s,t) = \delta_{ij}(1 - {}^{(n)}H_i(s,t)) + \sum_{k \in I} \int_s^t {}^{(n)}\phi_{kj}(u,t)\, {}^{(n-1)}Q_{ik}(s,du), i, j \in I. \quad (21.18)$$

From relation (21.1), it is clear that we have:

$$P(J_n = j \,|\, (J_k, T_k), k \leq n-1) = {}^{(n-1)} p_{J_{n-1}j}(T_{n-1}), a.s. \quad (21.19)$$

It follows that the process $(J_n, n \geq 0)$ can be viewed as a *conditional multiple Markov chain*; this means that, given the sequence $(T_n, n \geq 0)$, each transition from $J_{n-1} \to J_n$ obeys a non-homogeneous Markov chain of kernel ${}^{(n-1)}P(T_{n-1})$ (see **Definition 21.3**).

Definition 21.9 *The conditional multiple Markov chain* $(J_n, n \geq 0)$ *is called the imbedded multiple MC.*

21.1.2 Special Cases

Let us point out that **Definition 21.2** is quite general as indeed it is non-homogeneous both for the time *s* and for the number of transitions *n*, this last one giving the possibility to model *epidemiological* phenomena such as AIDS for example (see in Janssen and Manca (2006)) the example of *Polya processes* and semi-Markov extensions).
This extreme generality gives importance to the following particular cases.

(i) Non-Homogeneous Markov Additive Process And Semi-Markov Process

If in the sequel $\mathbf{Q} = \left({}^{(n-1)}\mathbf{Q}(s,t), n \geq 1\right)$, we have:

$${}^{(n-1)}\mathbf{Q}(s,t) = \mathbf{Q}(s,t), n \geq 1, s < t, \quad (21.20)$$

that is $\mathbf{Q}$ independent of *n*, then the kernel $\mathbf{Q}$ is called a non-homogeneous semi-Markov kernel (in short NHSMK) defining a non-homogeneous Markov additive process (in short NHMAP) $((J_n, T_n), n \geq 0)$ and a non-homogeneous semi-Markov process (in short NHSMP) $Z = (Z(t), t \geq 0)$.
This family was introduced in a different way by Hoem (1972).
It is clear that the relation (21.20) means that the sequences

$${}^{(n-1)}\mathbf{F}(s,t) = \mathbf{F}(s,t), \; {}^{(n-1)}\mathbf{P}(s) = \mathbf{P}(s) \quad \forall n \geq 0 \quad (21.21)$$

are independent of *n* or equivalently that

$$\mathbf{Q}(s,t) = \mathbf{P}(s) \cdot \mathbf{F}(s,t). \quad (21.22)$$

Let us point out that, in this case, relations (21.18) become:

$$\phi_{ij}(s,t) = \delta_{ij}(1 - H_i(s,t)) + \sum_k \int_s^t \phi_{ij}(u,t) Q_{ik}(s,du), i, j \in I. \quad (21.23)$$

If moreover, we have

$$\mathbf{P}(s) = \mathbf{P}, s \geq 0, \tag{21.24}$$

then the kernel **Q** is called a *partially non-homogeneous semi-Markov kernel* (in short PNHSMK) defining a *partially non-homogeneous Markov additive process* (in short PNHMAP) $\left((J_n, T_n), n \geq 0\right)$ and a partially non-homogeneous semi-Markov process (in short PNHSMP) $Z = (Z(t), t \geq 0)$.
This family was introduced in a different way by Hoem (1972).

(ii) Non-Homogeneous MC

If the sequences ${}^{(n-1)}\mathbf{P}(s), \forall s \geq 0$ are independent of *s*, then $\left(J_n, n \geq 0\right)$ is a classical *non-homogeneous MC* (in short NHMC)

(iii) Homogeneous Markov Additive Process

A PNHSMK **Q** such that

$$F(s,t) = F(t-s), s,t \geq 0, t-s \geq 0, \tag{21.25}$$

is of course a classical homogeneous SM kernel as in section 2.

(iv) Non-Homogeneous Renewal Process

For *m=1*, The CNHMRP of kernel **Q** is given by

$$\mathbf{Q}(s,t) = ({}^{(n-1)}\mathbf{F}(s,t)), s,t > 0, t-s \geq 0 \tag{21.26}$$

and characterizes the sequence $(X_n, n \geq 0)$ with, as in (21.1),

$$\begin{aligned} &X_0 = 0, a.s., \\ &P(X_n \leq x \mid X_k, k \leq n-1) = {}^{(n-1)}F(T_{n-1}, T_{n-1} + x), x \in \mathbb{R}^+, \\ &T_0 = 0, T_n = \sum_{k=0}^{n} X_k, a.s. \end{aligned} \tag{21.27}$$

In this case, the process $(X_n, n \geq 0)$ is called a *completely non-homogeneous dependent renewal process* (in short CNHDRP) of kernel **Q**.
If moreover,

$${}^{(n-1)}F(s,t) = {}^{(n-1)}F(t-s), s,t > 0, t-s \geq 0, n \geq 1, \tag{21.28}$$

it follows that

$$\begin{aligned} &X_0 = 0, a.s., \\ &P(X_n \leq x \mid X_k, k \leq n-1) = {}^{(n-1)}F(x), x \in \mathbb{R}^+, n \geq 1 \end{aligned} \tag{21.29}$$

and so the process $(X_n, n \geq 0)$ is a sequence of *t* independent r.v. called a *completely non-homogeneous renewal process* (in short CNHRP) of kernel *F*.

Remark 21.1 In the non-homogeneous case, it is much more difficult to obtain asymptotic results (see for example Benevento (1986), Thorisson (1986), Papadopoulou-Vassiliou (1994)) for interesting theoretical results). That is not so dramatic as that we can say non-homogeneous models are used for modelling *transient* situations and not *asymptotic* ones and that is why we personally think that all attention must be given to the construction of numerical methods for example to be able to solve the non-homogeneous integral equations system (21.23); this is done in the next chapter.
However, let us mention that, for the particular case of non-homogeneous Markov chains, there exist more asymptotic results (see for example Isaacson and Madsen (1976).

Chapter 4

DISCRETE TIME AND REWARD SMP AND THEIR NUMERICAL TREATMENT

1 DISCRETE TIME SEMI-MARKOV PROCESSES

1.1 Purpose

This chapter will present both discrete time homogeneous (DTHSMP) and non-homogeneous (DTNHSMP) semi-Markov processes and the numerical methods to be used for applying semi-Markov models in real-life problems, furthermore the Semi-Markov ReWard Processes (SMRWP) will be presented.
Although, in general, time in real-life problems is continuous, the real observation of the considered system is almost always made in discrete time even if the used time unit may in some cases be very small.
The choice of this time unit depends on what we observe and what we wish to study.
For example if we are studying the random evolution of the earthquake activity in a tectonic fracture zone, then it could be observed with a unitary time scale of ten years. If we are studying the behaviour of a disablement resulting from a job related illness, the unitary time could be one year, and so on.
So it results that the phenomenon of time evolution is continuous, nevertheless usually, the observations are discrete in time.
Consequently, if we construct a model to be fitted with real data, in our opinion, it would be better to begin with discrete time models.

1.2 DTSMP Definition

Though DTHSMP and DTNHSMP definitions are similar to the continuous ones, for the sake of completeness, we will recall these definitions using directly the terminology used for discrete time models.
Let $I=\{1, 2, \ldots, m\}$ be the state space and let $\{\Omega, \Im, P\}$ be a probability space. Let us also define the following random variables:

$$J_n : \Omega \to I, \qquad T_n : \Omega \to \mathbb{N}. \tag{1.1}$$

Definition 1.1 *The process* (J_n, T_n) *is a discrete time homogeneous Markov renewal process or a discrete time non-homogeneous Markov renewal process if* *the kernels* **Q** *associated with the process are defined respectively in the following way*:

$$\mathbf{Q} = \left[Q_{ij}(t) \right] = \left[P\left(J_{n+1} = j,\ T_{n+1} - T_n \le t \mid J_n = i \right) \right] \quad i, j \in I,\ t \in \mathbb{N}, \quad (1.2)$$

$$\mathbf{Q} = \left[Q_{ij}(s,t) \right] = \left[P\left(J_{n+1} = j,\ T_{n+1} \le t \mid J_n = i, T_n = s \right) \right] i, j \in I,\ s,t \in \mathbb{N}. (1.3)$$

As in the continuous time case, it results that for the homogeneous case, we define:

$$\mathbf{P} = \left[p_{ij} \right] = \left[\lim_{t\to\infty} Q_{ij}(t) \right];\ i, j \in I,\ t \in \mathbb{N}. \quad (1.4)$$

For the non-homogeneous case, we obtain:

$$\mathbf{P} = \left[p_{ij}(s) \right] = \left[\lim_{t\to\infty} Q_{ij}(s,t) \right];\ i, j \in I,\ s,t \in \mathbb{N}, \quad (1.5)$$

P being the transition matrix of the *embedded Markov chain* of the process.
Furthermore it is necessary to introduce the probability that the process will leave the state i before or at a time t:

$$\mathbf{H} = \left[H_i(t) \right] = \left[P\left(T_{n+1} - T_n \le t \mid J_n = i \right) \right], \quad (1.6)$$

$$\mathbf{H} = \left[H_i(s,t) \right] = \left[P\left(T_{n+1} \le t \mid J_n = i, T_n = s \right) \right]. \quad (1.7)$$

From the results of Chapter 3, we know that obviously:

$$H_i(t) = \sum_{j=1}^{m} Q_{ij}(t) \text{ and } H_i(s,t) = \sum_{j=1}^{m} Q_{ij}(s,t). \quad (1.8)$$

The following probabilities only have sense in the discrete time case and to be concise, we present first the definition for the homogeneous case and then for the non-homogeneous one.

Definition 1.2 *Matrix* **B** *is defined as follows:*

$$\mathbf{B} = \left[b_{ij}(t) \right] = \left[P\left(J_{n+1} = j,\ T_{n+1} - T_n = t \mid J_n = i \right) \right], \quad (1.9)$$

$$\mathbf{B} = \left[b_{ij}(s,t) \right] = \left[P\left(J_{n+1} = j,\ T_{n+1} = t \mid J_n = i, T_n = s \right) \right]. \quad (1.10)$$

From **Definition 1.1** it results that:

$$b_{ij}(t) = \begin{cases} Q_{ij}(0) = 0 & \text{if} \quad t = 0, \\ Q_{ij}(t) - Q_{ij}(t-1) & \text{if} \quad t = 1,2,..., \end{cases} \quad (1.11)$$

$$b_{ij}(s,t) = \begin{cases} Q_{ij}(s,s) = 0 & \text{if} \quad t = s, \\ Q_{ij}(s,t) - Q_{ij}(s,t-1) & \text{if} \quad t > s. \end{cases} \quad (1.12)$$

Definition 1.3 *The discrete time conditional distribution functions of the waiting times given the present and the next states, are given by:*

$$\mathbf{F} = \left[F_{ij}(t) \right] = \left[P\left(T_{n+1} - T_n \le t \mid J_n = i, J_{n+1} = j \right) \right], \quad (1.13)$$

$$\mathbf{F} = \left[F_{ij}(s,t) \right] = \left[P\left(T_{n+1} \le t \mid J_n = i, J_{n+1} = j, T_n = s \right) \right]. \quad (1.14)$$

Obviously the related probabilities can be obtained by means of the following formulas:

$$F_{ij}(t) = \begin{cases} Q_{ij}(t)/p_{ij} & \text{if} \quad p_{ij} \neq 0, \\ U_1(t) & \text{if} \quad p_{ij} = 0, \end{cases} \tag{1.15}$$

$$F_{ij}(s,t) = \begin{cases} Q_{ij}(s,t)/p_{ij}(s) & \text{if} \quad p_{ij}(s) \neq 0, \\ U_1(s,t) & \text{if} \quad p_{ij}(s) = 0, \end{cases} \tag{1.16}$$

where $U_1(t) = U_1(s,t) = 1 \ \forall s,t$.

Now, we can introduce the *discrete time semi-Markov process* $Z = (Z(t), t \in \mathbb{N})$ where $Z(t) = J_{N(t)}$, $N(t) = \max\{n : T_n \leq t\}$ representing the state occupied by the process at time *t*.

For *i,j=1,…,m*, the transition probabilities are defined in the following way:

$$\phi_{ij}(t) = \mathrm{P}(Z_t = j \mid Z_0 = i) \tag{1.17}$$

for the homogeneous case; for the non-homogeneous case, we have:

$$\phi_{ij}(s,t) = \mathrm{P}(Z_t = j \mid Z_s = i). \tag{1.18}$$

They are obtained by solving the following evolution equations:

$$\phi_{ij}(t) = \delta_{ij}(1 - H_i(t)) + \sum_{\beta=1}^{m}\sum_{\vartheta=1}^{t} b_{i\beta}(\vartheta)\phi_{\beta j}(t-\vartheta), \tag{1.19}$$

$$\phi_{ij}(s,t) = \delta_{ij}(1 - H_i(s,t)) + \sum_{\beta=1}^{m}\sum_{\vartheta=s+1}^{t} b_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t), \tag{1.20}$$

where, as usual, δ_{ij} represents the Kronecker symbol.

2 NUMERICAL TREATMENT OF SMP

In this section, we present the numerical solutions of the evolution equation of continuous time semi-Markov process in homogeneous and non-homogeneous cases.

The proposed approach uses a general quadrature method and we will prove that the numerical solution tends to the solution of the evolution equation of the continuous time HSMP.

It will also be shown that, using a very particular quadrature formula for the numerical solution of evolution equations of continuous time processes, it is possible to obtain the evolution equation of discrete time processes.

These results were obtained in Corradi et al (2004) and Janssen-Manca (2001a) generalizing the classical results on integral equation numerical solutions (see Baker (1977)).

Let us consider a continuous time SMP of kernel **Q** supposed to be differentiable.

First of all we write down the evolution equations of the SMP (see Chapter 3, relation (10.2) and (21.18)) as follows:

$$\phi_{ij}(t) \;=\; \delta_{ij}(1-H_i(t))+\sum_{\beta=1}^{m}\int_0^t \dot{Q}_{i\beta}(\vartheta)\phi_{\beta j}(t-\vartheta)d\vartheta\,, \tag{2.1}$$

$$\phi_{ij}(s,t) \;=\; \delta_{ij}(1-H_i(s,t))+\sum_{\beta=1}^{m}\int_s^t \dot{Q}_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t)d\vartheta\,, \tag{2.2}$$

where $\dot{Q}_{ij}$ represents the derivative respect to time of Q_{ij}.

Each generic quadrature formula can be written as (see Evans (1993)):

$$\int_0^{kh} f(t)dt \cong \sum_{l=0}^{k} w_{k,l}f(lh), \tag{2.3}$$

where h is the step length, $k \le N,\quad k,N \in \mathbb{N},\quad w_{k,l}$ are the weights related to the quadrature formula (10.1); they are functions of both the end point and the point in which we compute the function value.

If we set:

$$d_{ij}(t) = (1-H_{ij}(t))\delta_{ij}, \tag{2.4}$$

$$d_{ij}(s,t) = (1-H_{ij}(s,t))\delta_{ij}, \tag{2.5}$$

we obtain respectively:

$$\tilde{\phi}_{ij}(kh) = d_{ij}(kh)+\sum_{l=1}^{m}\left(\sum_{\tau=0}^{k} w_{k\tau}\tilde{\phi}_{lj}(kh-\tau h)\dot{Q}_{il}(\tau h)\right), \tag{2.6}$$

$$\tilde{\phi}_{ij}(uh,kh) = d_{ij}(uh,kh)+\sum_{l=1}^{m}\left(\sum_{\tau=u}^{k} w_{u,k,\tau}\tilde{\phi}_{lj}(\tau h,kh)\dot{Q}_{il}(uh,\tau h)\right), \tag{2.7}$$

where h is the step length, w the weights related to the quadrature formulas, $0 \le u \le k \le N,\ u,k,N \in \mathbb{N},$ such that $Nh = Y$ and $[0,Y]$ is the integration interval.

Now we proceed showing only the homogeneous case but all the results given for the homogeneous case hold in both cases. The reader interested in acquiring more details can refer to Janssen and Manca (2001a) and Corradi et al (2004).

We suppose we have already computed:

$$\begin{matrix} \phi_{1j}(0) & \phi_{1j}(h) & \phi_{1j}(2h) & \cdots & \phi_{1j}((k-1)h) \\ \phi_{2j}(0) & \phi_{2j}(h) & \phi_{2j}(2h) & \cdots & \phi_{2j}((k-1)h) \\ \phi_{3j}(0) & \phi_{3j}(h) & \phi_{3j}(2h) & \cdots & \phi_{3j}((k-1)h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{mj}(0) & \phi_{mj}(h) & \phi_{mj}(2h) & \cdots & \phi_{mj}((k-1)h) \end{matrix} \tag{2.8}$$

where $1 \le k \le N$ and

$$\tilde{\phi}_{ij}(0) = \phi_{ij}(0) = (1-H_i(0))\delta_{ij},\quad i=1,\ldots,m,\quad j=1,\ldots,m\,. \tag{2.9}$$

Then from (2.6) it follows that for a fixed j and $\forall i = 1,\ldots,m$:

$$\tilde{\phi}_{ij}(kh)-\sum_{l=1}^{m} w_{k0}\tilde{\phi}_{lj}(kh)\dot{Q}_{il}(0) = d_{ij}(kh)+\sum_{l=1}^{m}\left(\sum_{\tau=1}^{k} w_{k\tau}\tilde{\phi}_{lj}(kh-\tau h)\dot{Q}_{il}(\tau h)\right). \tag{2.10}$$

From relation (2.8), we get

$$\begin{array}{rcl}\tilde{\phi}_{1j}(kh)-\sum_{l=1}^{m} w_{k0}\tilde{\phi}_{lj}(kh)\dot{Q}_{1l}(0) & = & c_1\\ \tilde{\phi}_{2j}(kh)-\sum_{l=1}^{m} w_{k0}\tilde{\phi}_{lj}(kh)\dot{Q}_{2l}(0) & = & c_2\\ \vdots & \vdots & \vdots\\ \tilde{\phi}_{mj}(kh)-\sum_{l=1}^{m} w_{k0}\tilde{\phi}_{lj}(kh)\dot{Q}_{ml}(0) & = & c_m\end{array} \tag{2.11}$$

where, for convenience, the $c_i,\ i=1,\dots,m$ represent the second member of (2.10).

The linear system (2.11) in the unknowns $\tilde{\phi}_{ij}(kh),\ i=1,\dots,m$ admits solution if the coefficient matrix is non-singular.

The following theorem holds:

Theorem 2.1 *Assume that*

$$Q_{ij}:[0,Y]\to\mathbb{R},\ \phi_{ij}:[0,Y]\to\mathbb{R} \tag{2.12}$$

and $q\in\{1,\dots,N\},\quad N\in\mathbb{N}$, *such that* $Nh\le Y$.

Furthermore let

$$\xi_{ij}^{k}(h)=\tilde{\phi}_{ij}(kh)-\phi_{ij}(kh),\qquad k=0,1,2,\dots,N \tag{2.13}$$

where $\phi_{ij}(kh)$ *is the solution of* (2.1) *and* $\tilde{\phi}_{ij}(kh)$ *is the solution of* (2.6) *and*

$$\eta^{k}(h)=\sum_{i=1}^{m}\left|\xi_{ij}^{k}(h)\right|. \tag{2.14}$$

Furthermore, let:

$$w=w_N=\max_{0\le u\le k\le N}\frac{|w_{ku}|}{h}<\infty, \tag{2.15}$$

$$t_{ij}^{k}(h)=\sum_{l=1}^{m}\int_0^{kh}\phi_{lj}(kh-\tau)\dot{Q}_{il}(\tau)d\tau-\sum_{l=1}^{m}\left[\sum_{u=0}^{k} w_{ku}\tilde{\phi}_{lj}(kh-uh)\dot{Q}_{il}(uh)\right], \tag{2.16}$$

$$\sigma^{k}(h)=\sum_{i=1}^{m}\left|t_{ij}^{k}(h)\right|, \tag{2.17}$$

$$\tau(h)=\tau_N(h)=\max_{q\le h\le N}\sigma^{k}(h), \tag{2.18}$$

$$\xi(h)=\sum_{u=0}^{q-1}\eta^{u}(h). \tag{2.19}$$

Then, if $\left|\dot{Q}_{ij}(t)\right|\le c_1$ *for* $t\in[0,Y]$, *we have:*

$$\eta^k(h) \le \frac{\tau(h) + mhw_N c_1 \xi(h)}{1 - mhw_N c_1} \exp\left(\frac{mw_N c_1\, kh}{1 - mhw_N c_1} \right), k = q, q+1, \ldots, N \quad (2.20)$$

given that $m, h, w_N, c_1 < 1$

Proof This theorem is a particular case of the one given in Janssen and Manca (2001a). □

Remark 2.1 In general, equation (2.6) cannot be solved exactly. Then if the values $\hat{\phi}_{ij}(kh)$ give the approximate solution of (2.6) (for $k \ge 0$), then

$$\hat{\phi}_{ij}(kh) = d_{ij}(kh) + \sum_{l=1}^{m} \left(\sum_{\tau=0}^{k} w_{k\tau} \hat{\phi}_{lj}(kh - \tau h) \dot{Q}_{il}(\tau h) \right) + \varsigma_{ij}^k, \quad (2.21)$$

if we suppose that: $\hat{\phi}_{ij}(0) = \tilde{\phi}_{ij}(0) = d_{ij}(0)$.

Then setting:

$$\pi_{ij}^k(h) = \hat{\phi}_{ij}(kh) - \tilde{\phi}_{ij}(kh) \quad (2.22)$$

it follows that

$$\pi_{ij}^0(h) = \hat{\phi}_{ij}(0) - \tilde{\phi}_{ij}(0), \qquad \forall i, j \in \{1, \ldots, m\}. \quad (2.23)$$

Using relations (2.6) (2.21) and (2.22), we get for $k \ge 1$,

$$\pi_{ij}^k(h) = \sum_{l=1}^{m} \left(\sum_{\tau=0}^{k} w_{k\tau} \dot{Q}_{il}(\tau h) \pi_{lj}^{k-\tau}(h) \right) + \varsigma_{ij}^k. \quad (2.24)$$

Using (2.15) and summing up with respect to the first index, it results that:

$$\sum_{l=1}^{m} \left| \pi_{lj}^k(h) \right| \le m \quad w_N \quad h \quad c_1 \sum_{l=1}^{m} \sum_{\tau=0}^{k} \left| \pi_{lj}^\tau(h) \right| + \sum_{l=1}^{m} \left| \varsigma_{ij}^k \right|. \quad (2.25)$$

If we set:

$$\overline{\eta}^k(h) = \sum_{l=1}^{m} \left| \pi_{lj}^k(h) \right|, \text{ and } \rho^k = \sum_{l=1}^{m} \left| \varsigma_{ij}^k \right|, \quad (2.26)$$

it follows that:

$$\overline{\eta}^k(h) \le \rho^k + mw_N hc_1 \sum_{\tau=0}^{k} \overline{\eta}^\tau(h). \quad (2.27)$$

Given the following positions:

$$\overline{\tau}_N = \max_{1 \le k \le N} \rho^k, \qquad \overline{\eta}^k(h) \le \overline{\tau}_N + mw_N hc_1 \sum_{\tau=0}^{k} \overline{\eta}^\tau(h), \quad (2.28)$$

that is:

$$(1 - mw_N hc_1)\overline{\eta}^k(h) \le \overline{\tau}_N + mw_N hc_1 \sum_{u=0}^{k-1} \overline{\eta}^u(h), \quad (2.29)$$

it follows finally that:

$$\eta^k(h) \leq \frac{\overline{\tau}_N}{1 - m\,h\,w_N\,c_1} \exp\left(\frac{m\,w_N\,c_1\;k\,h}{1 - m\,h\,w_N\,c_1} \right), \qquad k = 1,\ldots,N. \qquad (2.30)\ \square$$

Remark 2.2 In the non-homogeneous case, as s is fixed in the system of integral equations (2.2), τ is the only parameter, so for each $s \in [0, Y]$ the result of **Theorem 2.1** holds.

3. DTSMP AND SMP NUMERICAL SOLUTIONS

In the previous section, we gave general formulas for the discretization of continuous time HSMP and NHSMP with a finite number of states. With the most simple quadrature method (rectangle formula), we get:

$$\tilde{\phi}_{ij}(kh) = d_{ij}(kh) + h\sum_{l=1}^{m}\left(\sum_{\tau=1}^{k} \tilde{\phi}_{lj}(kh - \tau h)\dot{Q}_{il}(\tau h) \right), \qquad (3.1)$$

$$\tilde{\phi}_{ij}(uh,kh) = d_{ij}(uh,kh) + h\sum_{l=1}^{m}\left(\sum_{\tau=u+1}^{k} \tilde{\phi}_{lj}(\tau h,kh)\dot{Q}_{il}(uh,\tau h) \right). \qquad (3.2)$$

Here, the sum on the time starts from 1 (u+1) as, the probability of changing state with a waiting time 0 is 0. Substituting in relations (3.1) and (3.2) the differential with the increment and with h=1, it results that:

$$\tilde{\phi}_{ij}(k) \cong d_{ij}(k) + \sum_{l=1}^{m}\left(\sum_{\tau=1}^{k} \tilde{\phi}_{lj}(k - \tau)\left(Q_{il}(\tau) - Q_{il}(\tau - 1) \right) \right), \qquad (3.3)$$

$$\tilde{\phi}_{ij}(u,k) \cong d_{ij}(u,k) + \sum_{l=1}^{m}\left(\sum_{\tau=u+1}^{k} \tilde{\phi}_{lj}(\tau,k)\left(Q_{il}(u,\tau) - Q_{il}(u,\tau - 1) \right) \right). \qquad (3.4)$$

Furthermore, taking into account relations (1.9) and (1.10) it results that

$$\tilde{\phi}_{ij}(k) \cong d_{ij}(k) + \sum_{l=1}^{m}\sum_{\tau=1}^{k} \tilde{\phi}_{lj}(k - \tau) b_{il}(\tau), \qquad (3.5)$$

$$\tilde{\phi}_{ij}(u,k) \cong d_{ij}(u,k) + \sum_{l=1}^{m}\sum_{\tau=u+1}^{k} \tilde{\phi}_{lj}(\tau,k) b_{il}(u,\tau). \qquad (3.6)$$

In this way, the evolution equations of the (DTHSMP) and (DTNHSMP) as defined in relations (1.19) and (1.20) are obtained:

$$\phi_{ij}(k) = d_{ij}(k) + \sum_{l=1}^{m}\sum_{\tau=1}^{k} \phi_{lj}(k - \tau) b_{il}(\tau), \qquad (3.7)$$

$$\phi_{ij}(u,k) = d_{ij}(u,k) + \sum_{l=1}^{m}\sum_{\tau=u+1}^{k} \phi_{lj}(\tau,k) b_{il}(u,\tau). \qquad (3.8)$$

If the discretization step is h, then relations (3.7) and (3.8) become:

$$\phi_{ij}^h(kh) = d_{ij}^h(kh) + \sum_{l=1}^{m}\sum_{\tau=1}^{k} b_{il}^h(\tau h) \quad \phi_{lj}^h((k - \tau)h), \qquad (3.9)$$

$$\phi_{ij}^h(uh,kh) = d_{ij}^h(uh,kh) + \sum_{l=1}^{m}\sum_{\tau=u+1}^{k} b_{il}^h(uh,\tau h) \quad \phi_{lj}^h(\tau h,kh). \tag{3.10}$$

Now the equations (3.9) and (3.10) can be rewritten in matrix form as follows:

$$\mathbf{\Phi}^h(kh) = \mathbf{D}^h(kh) + \sum_{\tau=1}^{k}\mathbf{B}^h(\tau h) * \mathbf{\Phi}^h((k-\tau)h), \tag{3.11}$$

$$\mathbf{\Phi}^h(uh,kh) = \mathbf{D}^h(uh,kh) + \sum_{\tau=u+1}^{k}\mathbf{B}^h(uh,\tau h) * \mathbf{\Phi}^h(\tau h,kh), \tag{3.12}$$

or equivalently:

$$\mathbf{\Phi}^h(kh) - \sum_{\tau=1}^{k}\mathbf{B}^h(\tau h) * \mathbf{\Phi}^h((k-\tau)h) = \mathbf{D}^h(kh), \tag{3.13}$$

$$\mathbf{\Phi}^h(uh,kh) - \sum_{\tau=u+1}^{k}\mathbf{B}^h(\tau h,kh) * \mathbf{\Phi}^h(\tau h,kh) = \mathbf{D}^h(uh,kh),\ k \in \mathbb{N}, u \le k. \tag{3.14}$$

Taking into account that $k \in \mathbb{N}$, both equations (3.13) and (3.14) can be written more compactly as:

$$\mathrm{U}^h * \mathbf{\Phi}^h = \mathbf{D}^h. \tag{3.15}$$

For the homogeneous case, it results that:

$$\mathbf{U}^h = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ -\mathbf{B}^h(h) & \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots \\ -\mathbf{B}^h(2h) & -\mathbf{B}^h(h) & \mathbf{I} & \mathbf{0} & \cdots \\ -\mathbf{B}^h(3h) & -\mathbf{B}^h(2h) & -\mathbf{B}^h(h) & \mathbf{I} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.16}$$

$$\mathbf{\Phi}^h = \begin{bmatrix} \mathbf{\Phi}^h(0) \\ \mathbf{\Phi}^h(h) \\ \mathbf{\Phi}^h(2h) \\ \mathbf{\Phi}^h(3h) \\ \vdots \end{bmatrix}, \quad \mathbf{D}^h = \begin{bmatrix} \mathbf{D}^h(0) \\ \mathbf{D}^h(h) \\ \mathbf{D}^h(2h) \\ \mathbf{D}^h(3h) \\ \vdots \end{bmatrix}$$

and in the non-homogeneous case:

$$\mathbf{U}^h = \begin{bmatrix} \mathbf{I} & -\mathbf{B}^h(0,h) & -\mathbf{B}^h(0,2h) & -\mathbf{B}^h(0,3h) & \cdots \\ \mathbf{0} & \mathbf{I} & -\mathbf{B}^h(h,2h) & -\mathbf{B}^h(h,3h) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{B}^h(2h,3h) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.17}$$

$$\mathbf{D}^h = \begin{bmatrix} \mathbf{I} & \mathbf{D}^h(0,h) & \mathbf{D}^h(0,2h) & \mathbf{D}^h(0,3h) & \cdots \\ \mathbf{0} & \mathbf{I} & \mathbf{D}^h(h,2h) & \mathbf{D}^h(h,3h) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{D}^h(2h,3h) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The following theorem holds for both homogeneous and non-homogeneous cases.

Theorem 3.1 *Equations* (3.13) *and* (3.14) *admit a unique solution.*

Proof The determinant of matrix $\mathbf{U}^h$ is absolutely convergent (Riesz (1913)); more precisely $\det(\mathbf{U}^h) = 1$ and consequently matrix $\mathbf{U}^h$ is invertible. □

In the homogenous case equation (3.13) can be seen as an infinite linear system with an infinite number of unknowns. It ensues from **Theorem 3.1** that the system is solvable.
To solve such a system it is usually necessary to apply the truncation method Riesz (1913); but in our case, it is very simple to find the solution.
Obviously, $\mathbf{\Phi}^h(0) = \mathbf{I}$ and, once $\mathbf{\Phi}^h(0)$ is known we get:

$$\mathbf{\Phi}^h(h) = \mathbf{B}^h(h) * \mathbf{\Phi}^h(0) + \mathbf{D}^h(h). \tag{3.18}$$

Once $\mathbf{\Phi}^h(0), \mathbf{\Phi}^h(h), \ \ldots, \ \mathbf{\Phi}^h(kh)$ are known, then:

$$\mathbf{\Phi}^h((k+1)h) = \sum_{\tau=1}^{k+1} \mathbf{B}^h(\tau h) * \mathbf{\Phi}^h((k+1-\tau)h) + \mathbf{D}^h((k+1)h) \tag{3.19}$$

and it is not necessary to apply the truncation method. Furthermore, these results are obtained without any matrix inversion.
As in the homogeneous case, equation (3.14) can be seen as an infinite linear system with an infinite number of unknowns, but the non-homogeneous matrix equation is different from the homogeneous one because the coefficient matrix is upper triangular in the non-homogeneous case and lower triangular in the homogeneous.
Also in this case the truncation method should not be applied. In the homogeneous case this result is trivial. In the non-homogeneous case the result is not so immediate. We report, with more precision, the following result given in and Janssen and Manca (2001a).

Proposition 3.1 *The solution of the infinite order linear system* (3.14) *can be given explicitly step by step.*

Proof For the proof see Janssen-Manca (2001a) □

Remark 3.1 Here it should also be mentioned that the upper triangularity of the coefficient block matrix and the fact that the matrices on the main diagonal are identity matrices implies that the results are obtained with no matrix inversion.
The following two theorems hold for both the homogeneous and non-homogeneous cases but only the homogeneous results will be given.

A straightforward proof of the randomness of the matrix $\mathbf{\Phi}^h$ is possible by exploiting the following:

Theorem 3.2 *The matrices* $\mathbf{\Phi}^h(kh)$ *are stochastic.*

Proof The result is true for $\mathbf{\Phi}^h(0) = \mathbf{I}$. We suppose that it is true for $\tau = 1,\dots, k$. We have to check what happens under these hypotheses for $k+1$.
From (1.11) the equation (3.19) becomes:

$$\begin{aligned}\mathbf{\Phi}^h((k+1)h) = &\sum_{\tau=1}^{k+1}\left(\mathbf{Q}^h(\tau h) - \mathbf{Q}^h((\tau-1)h)\right)\mathbf{\Phi}^h((k+1-\tau)h) \\ &+ \mathbf{D}^h((k+1)h).\end{aligned} \tag{3.20}$$

To prove that $\mathbf{\Phi}^h((k+1)h)$ is stochastic we have to show that:

$$\sum_{j=1}^{m}\phi_{ij}^h((k+1)h) = 1, \quad i = 1,\dots, m. \tag{3.21}$$

An element of (3.20) is given by:

$$\begin{aligned}\phi_{ij}^h((k+1)h) = &\, d_{ij}^h((k+1)h) + \sum_{\tau=1}^{k+1}\sum_{l=1}^{m} Q_{il}^h(\tau h)\quad \phi_{lj}^h((k+1-\tau)h) \\ &-\sum_{\tau=1}^{k+1}\sum_{l=1}^{m} Q_{il}^h((\tau-1)h)\quad \phi_{lj}^h((k+1-\tau)h).\end{aligned} \tag{3.22}$$

Summing up with respect to j and taking into account relations (2.4), (1.8), (1.19) and the inductive hypothesis, the following result is obtained:

$$\begin{aligned}\sum_{j=1}^{m}\phi_{ij}^h((k+1)h) = &\, 1 - \sum_{j=1}^{m} Q_{ij}^h((k+1)h) + \sum_{\tau=1}^{k+1}\sum_{l=1}^{m} Q_{il}^h(\tau h)\sum_{j=1}^{m}\ \phi_{lj}^h((k+1-\tau)h) \\ &-\sum_{\tau=1}^{k+1}\sum_{l=1}^{m} Q_{il}^h((\tau-1)h)\sum_{j=1}^{m}\ \phi_{lj}^h((k+1-\tau)h) = 1.\end{aligned} \tag{3.23}$$

□

Now let Z be a continuous time HSMP with $\mathbf{\Phi}$ as evolution equation and $\{T_n\}$ as sequence of the state change times.
If we set:

$$T_n^h = \left\lfloor \frac{T_n}{h} \right\rfloor h \tag{3.24}$$

and

$$Z^h(t) = J_n \quad \text{if} \quad T_n^h \le t < T_{n+1}^h, \tag{3.25}$$

then Z^h is a DTHSMP with evolution equations given by:

$$\phi_{ij}^h(kh) = d_{ij}^h(kh) + \sum_{l=1}^{m}\sum_{\tau=1}^{k} b_{il}^h(\tau h)\phi_{lj}^h((k-\tau)h) \tag{3.26}$$

and it is defined on the same probability space $\{\Omega, \Im, P\}$ of Z.
Given $\omega \in \Omega$ the following result holds P-almost.

Theorem 3.3 *The* Z^h *process converges to* Z *for* $h \to 0$ *in the Skorohod topology*[1].

Proof It must be shown that $\forall T > 0$ there exists a time rescaling sequence $\{\lambda^h\}$ where λ^h is a continuous, strictly increasing and surjective function from $[0,+\infty)$ to $[0,+\infty)$ such that:

$$\sup_{t\le T}\left|\lambda^h(t) - t\right| \to 0 \quad \text{if} \quad h \to 0 \tag{3.27}$$

and

$$\sup_{t\le T}\left|Z^h\left(\lambda^h(t)\right) - Z(t)\right| \to 0 \quad \text{if} \quad h \to 0; \tag{3.28}$$

(see Ethier and Kurtz (1986)).
Obviously,

$$T_n \to +\infty \quad \text{if} \quad n \to +\infty, \tag{3.29}$$

then it is sufficient to verify that the proposition holds for $T = T_n, \forall n$.
If we set:

$$h < \min_{l\le k\le n}\left|T_k^h - T_{k-1}^h\right|, \qquad T_0^h = 0, \tag{3.30}$$

let λ^h be the linear function that transforms the intervals $\left[T_{k-1},\ T_k\right]$ into $\left[T_{k-1}^h,\ T_k^h\right]$ with $1 \le k \le n$ and $T_0 = 0$, given by:

$$\lambda^h(t) = T_h^n + (t - T_n), \qquad t \ge T_n. \tag{3.31}$$

Then $\{\lambda^h\}$ verifies the proposition's conditions and:

$$\left|Z^h(\lambda^h(t)) - Z(t)\right| = 0 \qquad \forall t \le T_n. \tag{3.32}$$

Then Z^h converges to Z in the Skorohod topology with probability 1 and therefore, in particular, Z^h converges in law, i.e. in the weak sense of stochastic processes. □

---

[1] Skorokhod topology is the topology defined on the set of trajectories of stochastic processes. For more details see (Billingsley (1968)).

Remark 3.2 The weak convergence of stochastic processes is in reality the weak convergence of their laws, where the laws are probability measures induced from the processes on the space of their trajectories. In this case it is the space of right continuous functions with left limits, in which the Skorokhod topology holds.

Finally it is to be observed that the λ^h sequence depends on T_n by means of (3.30).

4 SOLUTION OF DTHSMP AND DTNHSMP IN THE TRANSIENT CASE: A TRANSPORTATION EXAMPLE

4.1 Principle Of The Solution

Generally speaking it is clear that, for the purpose of application, it is more worthwhile to solve systems (3.13) and (3.14) in a finite time horizon, which means that the process is solved in the transient case.
In the following, an algorithm useful for solving both evolution equations is briefly presented.
First of all, an epoch *T* is fixed and, in this light, equation (3.15) for both cases becomes:

$$ {}^{T}\mathbf{U} * {}^{T}\mathbf{\Phi} = {}^{T}\mathbf{D} . \tag{4.1} $$

The algorithms solve the linear system (4.1), in the sense that for known matrices ${}^{T}\mathbf{F}$ and **P**, it determines the matrix of the unknown ${}^{T}\mathbf{\Phi}$ by means of two iterative procedures. For the algorithms it is not necessary to compute the **U** matrix, but it is enough to construct the matrices **B** whose elements are defined respectively in (1.11) and (1.12).
The variables involved in the algorithms are:

INPUTS:
m, T, **P** (embedded M.C.), ${}^{T}\mathbf{F}$ (Matrix of the increasing distribution function of waiting times)

RESULTS:
${}^{T}\mathbf{Q}$, ${}^{T}\mathbf{B}$, ${}^{T}\mathbf{S}$, ${}^{T}\mathbf{D}$, ${}^{T}\mathbf{\Phi}$

STEPS:
Reads m, T
Reads **P**
Reads ${}^{T}\mathbf{F}$

Constructs ${}^{T}\mathbf{Q}$
Constructs ${}^{T}\mathbf{B}$
Constructs ${}^{T}\mathbf{S}$
Constructs ${}^{T}\mathbf{D}$
Solves the system and finds ${}^{T}\mathbf{\Phi}$

The steps are very general and hold in both cases. As we already said, the main difference is given by the fact that in the homogeneous case the matrices are lower triangular and upper triangular in the non-homogeneous case, see (3.16) and (3.17).
For a complete description of algorithms respectively in homogeneous and non-homogeneous cases, we refer to Corradi et al (2004) and Janssen-Manca (2001a).

4.2 Semi-Markov Transportation Example

4.2.1 Homogeneous Case

In this first example of a semi-Markov model, we extend to a semi-Markov environment the transportation problem presented in section 9.6 of Chapter 2.
We consider that a taxicab driver will work for eight hours. So we will work in the transient case within 32 time periods, which means that a period is a time interval of 15 minutes. In this way we can consider the full working time of a driver.
In this model, $\phi_{ij}(t)$ represents the probability that a driver who is in the state i will be in the state j after a time t and we have to solve the evolution equation.
This example is really simple so all the steps that are necessary to solve the DTHSMP could be shown.
The input is:
$m = 3$,
$T = 32$.
The matrix **P** was given in the formula (9.108) of Chapter 2 but in real application it could be constructed by data in the following way.
We are supposed to know all the runs that were driven in one month by all the taxicab drivers, so we know for each run the starting zone, the arriving zone and the course duration.
We refer to our case study, so we have three states, and we should construct the transition matrix **P** by the known data. We could construct a matrix **A** where
a_{ij} = number of runs from the zone i to the zone j in the month.
Then, it follows that:

$$p_{ij} = \frac{a_{ij}}{\sum_{k}^{3} a_{ik}}. \tag{4.2}$$

The matrix $\mathbf{F} = \left[F_{ij}(t) \right]$, the discrete time increasing probability distribution of the waiting time in each state i given that the next state to be successively occupied is j, should be constructed by the data.
The way to construct these d.f. by the data is the following.
We would construct for each i and j the related d.f.

$$F_{ij}(0), F_{ij}(1), F_{ij}(2), \ldots, F_{ij}(32). \tag{4.3}$$

From our data we compute the vector

$$\mathbf{v}_{ij} = (v_{ij}(1), v_{ij}(2), \ldots, v_{ij}(32), v_{ij}(33)) \tag{4.4}$$

where $v_{ij}(1)$ represents the number of all the runs that have a duration (including also the waiting time of the taxi driver before beginning the run) less than or equal to 15 minutes, $v_{ij}(2)$ the number of all the courses that have a duration greater than 15 minutes and less than or equal to 30 minutes and so on.
In $v_{ij}(33)$ there will be the number of all the runs from i and j that have a duration larger than eight hours if any.
From the vector $\mathbf{v}_{ij}$ we can construct the vector $\mathbf{w}_{ij}$:

$$w_{ij}(t) = \sum_{s=1}^{t} v_{ij}(s) \qquad t = 1, \ldots, 33. \tag{4.5}$$

So, we finally get the elements of the matrix $\mathbf{F}$:

$$F_{ij}(0) = 0,\ F_{ij}(t) = \frac{w_{ij}(t)}{w_{ij}(33)}, \qquad t = 1, \ldots, 32. \tag{4.6}$$

To illustrate this method proposed for real data, we will here construct artifical data and find matrix **F** by means of pseudorandom generator numbers.
Our example is very simple (only three states) but with 32 time periods, reporting of all the matrices involved in the computation will be too long and so we will report for each matrix the time periods 1, 5, 10, 20, 30 and 32.
We get the following results:

Matrix **F**

F(1)			**F**(5)			**F**(10)		
0.0470	0.0396	0.0514	0.1835	0.1299	0.2349	0.2863	0.2581	0.3524
0.0200	0.0449	0.0009	0.1478	0.0847	0.0910	0.3083	0.2648	0.2924
0.0456	0.0168	0.0325	0.1314	0.1472	0.1106	0.2828	0.2491	0.3183
F(20)			**F**(30)			**F**(32)		
0.5288	0.5628	0.6047	0.8268	0.8377	0.9236	0.8468	0.8574	0.9938
0.5877	0.4767	0.5842	0.8943	0.7982	0.8997	0.9423	0.8272	0.9382
0.5500	0.4956	0.6617	0.8721	0.8131	0.8376	0.9307	0.8478	0.9028

Note that, always, **F**(0)=0 because there are no movements in a time 0, and **F**(32)<1 because it is the last time period and therefore a truncated d.f.
After the computation of matrix **F**, we can compute the kernel **Q** as follows (see relation (1.15))

$$\mathbf{Q}(t) = \mathbf{F}(t) \cdot \mathbf{P}, \tag{4.7}$$

where the symbol $\cdot$ means matrix product element by element.

Matrix **Q**

Q(1)			**Q**(5)			**Q**(10)		
0.0235	0.0158	0.0051	0.0918	0.0520	0.0235	0.1432	0.1032	0.0352
0.0060	0.0269	0.0001	0.0443	0.0508	0.0091	0.0925	0.1589	0.0292
0.0091	0.0017	0.0227	0.0263	0.0147	0.0775	0.0566	0.0249	0.2228
Q(20)			**Q**(30)			**Q**(32)		
0.2644	0.2251	0.0605	0.4134	0.3351	0.0924	0.4234	0.3430	0.0994
0.1763	0.2860	0.0584	0.2683	0.4789	0.0900	0.2827	0.4963	0.0938
0.1100	0.0496	0.4632	0.1744	0.0813	0.5863	0.1861	0.0848	0.6319

The next matrix to be computed is the matrix **B** using the following result:

$$\mathbf{B}(t) = \begin{cases} \mathbf{0} & \text{if} \quad t = 0, \\ \mathbf{Q}(t) - \mathbf{Q}(t-1) & \text{if} \quad t > 0. \end{cases} \tag{4.8}$$

Matrix **B**

B(1)			**B**(5)			**B**(10)		
0.0235	0.0158	0.0052	0.0194	0.0061	0.0046	0.0061	0.0079	0.0032
0.0060	0.0269	0.0001	0.0029	0.0153	0.0009	0.0095	0.0303	0.0040
0.0091	0.0017	0.0227	0.0110	0.0023	0.0257	0.0009	0.0006	0.0331
B(20)			**B**(30)			**B**(32)		
0.0222	0.0151	0.0000	0.0083	0.0097	0.0034	0.0062	0.0059	0.0053
0.0033	0.0138	0.0045	0.0067	0.0251	0.0048	0.0128	0.0047	0.0034
0.0038	0.0011	0.0020	0.0115	0.0045	0.0040	0.0078	0.0013	0.0240

Let us just mention that the 0 in position $b_{1,3}(20)$ is a numerical zero in the sense that rounding this number at the fourth decimal gives 0.
Then, we have to compute the estimation of matrix **H**, whose elements are the following, see relation (1.8):

$$H_{ij}(t) = \begin{cases} 0 & \text{if} \quad i \neq j, \\ \sum_{k=1}^{m} Q_{ik}(t) & \text{if} \quad i = j. \end{cases} \tag{4.9}$$

Matrix **H**

H(1)			**H**(5)			**H**(10)		
0.0445	0	0	0.1672	0	0	0.2816	0	0
0	0.0330	0	0	0.1043	0	0	0.2806	0
0	0	0.0335	0	0	0.1184	0	0	0.3043
H(20)			**H**(30)			**H**(32)		
0.5500	0	0	0.8408	0	0	0.8658	0	0
0	0.5207	0	0	0.8372	0	0	0.8728	0
0	0	0.6228	0	0	0.8421	0	0	0.9028

We know that these elements represent the probability to leave the state i in a period less than or equal to the period t, and so have sense only in the main diagonal of each submatrix.
The next matrix **D**,whose elements represent the probability of remaining in the state for t periods, is given by

$$\mathbf{D}(t) = \mathbf{I} - \mathbf{H}(t) . \tag{4.10}$$

We get:

Matrix **D**

D(1)			**D**(5)			**D**(10)		
0.9555	0	0	0.8328	0	0	0.7184	0	0
0	0.9670	0	0	0.8957	0	0	0.7194	0
0	0	0.9665	0	0	0.8816	0	0	0.6957
D(20)			**D**(30)			**D**(32)		
0.4500	0	0	0.1592	0	0	0.1342	0	0
0	0.4793	0	0	0.1628	0	0	0.1272	0
0	0	0.3772	0	0	0.1579	0	0	0.0972

The matrix we look for, that is $\mathbf{\Phi}$, is the solution of the evolution equation of the DTHSMP.
Here, $\phi_{ij}(t)$ represents the probability that a taxicab driver being at time 0 in zone i will be after t periods, in the state j.
From the results given below, any row of the submatrix $\mathbf{\Phi}(t)$ is indeed a probability distribution.
The results are:

Matrix $\mathbf{\Phi}$

$\mathbf{\Phi}$ (1)			$\mathbf{\Phi}$ (5)			$\mathbf{\Phi}$ (10)		
0.9790	0.0158	0.0052	0.9228	0.0530	0.0242	0.8553	0.1065	0.0382
0.0060	0.9939	0.0001	0.0437	0.9465	0.0098	0.0916	0.8767	0.0317
0.0091	0.0017	0.9892	0.0264	0.0154	0.9582	0.0582	0.0288	0.9130
$\mathbf{\Phi}$ (20)			$\mathbf{\Phi}$ (30)			$\mathbf{\Phi}$ (32)		
0.6973	0.2320	0.0707	0.5337	0.3490	0.1173	0.5132	0.3587	0.1281
0.1767	0.7548	0.0685	0.2745	0.6116	0.1139	0.2902	0.5882	0.1216
0.1220	0.0686	0.8094	0.2039	0.1278	0.6683	0.2196	0.1386	0.6418

4.2.2 Non-Homogeneous Case

As above in the homogoneous case, we consider that a taxicab driver will work for eight hours with time intervals of 15 minutes and so we will consider the transient case within 32 time periods for which $\phi_{ij}(s,t)$ represents the probability that a driver being in state *i* at time *s* will be in state *j* at time *t*.
Though this example is one of the simplest that can be done, it will clearly confirm that non-homogeneity, as already shown, gives some intrinsic supplementary difficulties.
Also, we will try to show all the steps that are necessary to understand the development of a DTNHSMP.
This time, the input is:
$m = 3$,
$T = 32$,
and the non-homogeneous Markov chain, reported in the following table.

P(0)			**P**(5)			**P**(10)		
0.3	0.4	0.3	0.39	0.35	0.26	0.49	0.3	0.21
0.4	0.2	0.4	0.35	0.32	0.33	0.3	0.42	0.28
0.32	0.38	0.3	0.28	0.33	0.39	0.23	0.28	0.49
P(20)			**P**(25)			**P**(29)		
0.54	0.35	0.11	0.5	0.4	0.1	0.5	0.4	0.1
0.3	0.62	0.08	0.3	0.6	0.1	0.3	0.6	0.1
0.23	0.18	0.59	0.24	0.13	0.63	0.2	0.1	0.7

Other input should be the matrix $\mathbf{F} = \left[F_{ij}(s,t) \right]$, the discrete time increasing probability distribution of the waiting time in each state *i*, given that the arrival time in the state *i* was at time *s* and that the next state to be successively occupied is *j*.
From the data, we can construct these distribution functions, as in the homogeneous cases.
To compute for each *s*, *i* and *j* the related d.f.,

$$F_{ij}(s,s), F_{ij}(s,s+1), F_{ij}(s,s+2), \ldots, F_{ij}(s,32), \quad s \le 32, \tag{4.11}$$

we first introduce the following quantities:

$$\mathbf{v}_{ij}(s) = (v_{ij}(s,s+1), v_{ij}(s,s+2), \ldots, v_{ij}(s,32), v_{ij}(s,33)). \tag{4.12}$$

$v_{ij}(s,s+1)$ gives the number of all the runs for which the taxi driver arrived at time *s* in the state *i* and finished the new course in the state *j* in a time less than or equal to 15 minutes (including the waiting time before beginning the new course).

Similarly $v_{ij}(s,s+2)$ gives the number of all the runs for which the taxi driver arrived at time s in the state i and finished the new run in the state j in a time more than 15 minutes and less than or equal to 30 minutes and so on.
Finally, $v_{ij}(s,32)$ gives the number of all the courses from i and j that began at the arrival time s and finished after 7.45 hours but within the eight hours of the turn and $v_{ij}(s,33)$ the number of taxi drivers who arrived at time s in i and who will finish the next course in j, but after the eight hours.
It is important to remark that in the semi-Markov environment the stopping time of the taxidriver before the beginning of another run is included in the duration of the course.
From the vector $\mathbf{v}_{ij}(s)$ we can construct the vector $\mathbf{w}_{ij}(s)$:

$$w_{ij}(s,t) = \sum_{h=1}^{t} v_{ij}(s,s+h) \qquad t = s+1,\ldots,33\,. \tag{4.13}$$

We finally obtain the elements of the matrix **F** given by:

$$F_{ij}(s,s) = 0, F_{ij}(s,t) = \frac{w_{ij}(s,t)}{w_{ij}(s,33)}, \qquad t = s+1,\ldots,32. \tag{4.14}$$

In place of real data not available here, we construct the **F** matrix by means of pseudorandom generator numbers as for the homogoneous case given above.
We can then multiply matrices **F** and **P** and obtain the matrix **Q** with the relation

$$\mathbf{Q}(s,t) = \mathbf{P}(s)\cdot\mathbf{F}(s,t)\,. \tag{4.15}$$

The next matrix to be computed is the matrix **B;** from the result:

$$\mathbf{B}(s,t) = \begin{cases} 0 & \text{if} \quad t \le s, \\ \mathbf{Q}(s,t) - \mathbf{Q}(s,t-1) & \text{if} \quad t > s, \end{cases} \tag{4.16}$$

we get:
Using the relation

$$H_{ij}(s,t) = \begin{cases} 0 & \text{if} \quad i \ne j, \\ \sum_{k=1}^{m} Q_{ik}(s,t) & \text{if} \quad i = j, \end{cases} \tag{4.17}$$

we finally obtain the matrix **H.**
The elements of this matrix represent the probability of leaving the state i in the time that goes from s to t. They have sense only on the main diagonal of each submatrix.
The matrix **D** representing the probabilites of remaining in the state from the time s up to the time t is obtained in the following way:

$$\mathbf{D}(s,t) = \mathbf{I} - \mathbf{H}(s,t)\,. \tag{4.18}$$

For the last step, we have to compute the matrix $\mathbf{\Phi}$, the solution of the evolution equation of the DTHSMP whose element $\phi_{ij}(s,t)$ represents the probability that a taxicab driver being at time s in the zone i will be at time t in state j.

Here, it can be verified that any row of the submatrix $\mathbf{\Phi}(s,t)$ is indeed a probability distribution.

Matrix $\mathbf{\Phi}$

$\mathbf{\Phi}(0,1)$			$\mathbf{\Phi}(0,10)$			$\mathbf{\Phi}(0,20)$		
0.9696	0.0255	0.0049	0.7936	0.1307	0.0757	0.6196	0.2089	0.1715
0.0123	0.9686	0.0191	0.1149	0.7845	0.1006	0.2473	0.5132	0.2395
0.0115	0.0046	0.9839	0.0862	0.1053	0.8085	0.2002	0.2142	0.5856
$\mathbf{\Phi}(5,6)$			$\mathbf{\Phi}(5,15)$			$\mathbf{\Phi}(5,25)$		
0.9898	0.0084	0.0018	0.7812	0.1290	0.0898	0.5840	0.2421	0.1739
0.0213	0.9685	0.0102	0.1374	0.7324	0.1302	0.2551	0.5364	0.2085
0.0164	0.0157	0.9679	0.1153	0.1147	0.7700	0.2069	0.2375	0.5556
$\mathbf{\Phi}(10,11)$			$\mathbf{\Phi}(10,21)$			$\mathbf{\Phi}(10,31)$		
0.9672	0.0242	0.0086	0.7275	0.1620	0.1105	0.5121	0.2971	0.1908
0.0201	0.9646	0.0153	0.1415	0.7095	0.1490	0.3241	0.4443	0.2316
0.0055	0.0057	0.9888	0.1254	0.1206	0.7540	0.2678	0.2726	0.4596
$\mathbf{\Phi}(20,21)$			$\mathbf{\Phi}(20,26)$			$\mathbf{\Phi}(20,31)$		
0.9892	0.0034	0.0074	0.7926	0.1583	0.0491	0.5561	0.3145	0.1294
0.0186	0.9718	0.0096	0.1277	0.8174	0.0549	0.2883	0.6075	0.1042
0.0289	0.0095	0.9616	0.1287	0.0781	0.7932	0.2420	0.2192	0.5388
$\mathbf{\Phi}(25,26)$			$\mathbf{\Phi}(25,28)$			$\mathbf{\Phi}(25,32)$		
0.9724	0.0096	0.0180	0.7794	0.1644	0.0562	0.4170	0.4360	0.1470
0.0715	0.9141	0.0144	0.1079	0.8313	0.0608	0.2924	0.5670	0.1406
0.0252	0.0009	0.9739	0.0552	0.0277	0.9171	0.2422	0.2194	0.5384
$\mathbf{\Phi}(29,30)$			$\mathbf{\Phi}(29,31)$			$\mathbf{\Phi}(29,32)$		
0.8975	0.0661	0.0364	0.7097	0.2161	0.0742	0.4500	0.4351	0.1149
0.0246	0.9532	0.0222	0.1911	0.7463	0.0626	0.2827	0.5960	0.1213
0.1264	0.0287	0.8449	0.1378	0.0702	0.7920	0.1824	0.1675	0.6501

In the non-homogeneous case we report only the final results, the interested reader can find them in the internet site given in the introduction.

5 CONTINUOUS AND DISCRETE TIME REWARD PROCESSES

In this part we will present undiscounted and discounted semi-Markov reward processes.
Reward processes can be seen as a class of stochastic processes. In the non-homogeneous case it is possible to write more than 200 different evolution equations of Semi-Markov ReWard Processes (SMRWP). We develop only three

cases, the simplest and the most general. For a wider approach the reader can refer to Janssen-Manca (2006).

5.1 Classification And Notation

5.1.1 Classification Of Reward Processes

In this book we will apply semi-Markov processes mainly in finance, insurance and reliability problems. In all these fields, the association of a sum of money to a state of the system and to a state transition assumes great relevance. In general, this can be done by attaching a reward structure to the process.
This structure can be seen as a random variable associated with the state occupancies and transitions (Howard (1971) vol. 2).
The rewards can be of different kinds, but in this book we will, because of our kind of applications, consider only amounts of money. These amounts can be positive, if for the system they can be seen as a *benefit* and negative if they can be considered as a *cost.*
The reward processes can be seen as classes of stochastic processes that we can classify in different cases. The following tables report the classification of the SMRWP.

Process classification

Homogeneous		
Non-Homogeneous		
Continuous time		
Discrete time		
Non-discounted		
Discounted	Fixed interest rate	
	Variable interest rate	Homogeneous interest law
		Non-homogeneous interest law

Reward classification

Time fixed rewards		
Time variable rewards	Homogeneous rewards	
	Non-homogeneous rewards	
Transition (impulse) rewards		
Permanence (rate) rewards	Discrete time	Immediate
		Due
	Independent on next transition	
	Dependent on next transition	

We will not present permanence rewards that depend on the next transition because in financial and insurance environments they do not have sense.
In general the distinction between homogeneous and non-homogeneous cases is done for stochastic processes. Also an interest rate law can be defined as homogeneous if the discount factor is a function only of the length of the financial operation, non-homogeneous if the discount factor takes into account also the initial time of operation, not only the duration.
In the same way, rewards can be fixed in time, can depend only on the duration or can be non-homogeneous in time.
We will use the following notation:
$\psi_i, \psi_i(t), \psi_i(s,t)$: represent rewards given for permanence in state *i;* the first is time fixed, the second changes because of time and the third represents a time non-homogeneous permanence reward.
$\gamma_{ij}, \gamma_{ij}(t), \gamma_{ij}(s,t)$: represent the three different kinds of rewards given for the transition from state *i* to state *j* (impulse reward).
In the discrete time case, the *immediate case* means that the reward is paid at the end of each period; in the *due case* the reward is paid at the beginning of the period. The impulse rewards γ represent lump sums that are paid at the transition instant.

5.1.2 Financial Parameters

To study the process with discounting, let us recall some basic results for computing present values of amounts of money, annuities and also the related notation.
For more details, refer to Volpe di Prignano (1985) or Kellison (1991).

Fixed time interest rate:
$v = (1+r)^{-1} = e^{-\delta}$: represents the one-period discount factor, where r is the interest rate and δ the corresponding intensity,
$a_{\overline{t}|r} = \dfrac{1-v^t}{r}$; $\ddot{a}_{\overline{t}|r} = \dfrac{1-v^t}{d}, d = \dfrac{r}{1+r}$; $\overline{a}_{\overline{t}|\delta} = \dfrac{1-e^{-\delta t}}{\delta}$ represent the present value of respectively a unitary annuity-immediate, an annuity-due and a continuous time annuity,
$a_{\overline{\infty}|r} = \dfrac{1}{r}, \ddot{a}_{\overline{\infty}|r} = \dfrac{1}{d}, \overline{a}_{\overline{\infty}|\delta} = \dfrac{1}{\delta}$ represent the present value of infinite time unitary annuities, also called *perpetuities.*

Time variable interest rate:
Now we suppose that the interest rates are variable and depend on the time period.:

$\nu(k) = \prod_{h=1}^{k}(1+r_h)^{-1}$, $\bar{\nu}(k) = e^{-\int_0^k \delta(\tau)d\tau}$ represent the *k-period discount factor* at time 0 respectively in discrete and continuous time. They give, at time 0, the discounted value of one monetary unit to be paid at the end of period *k,*

$\sum_{k=1}^{t}\nu(k)\psi_i(k), \sum_{k=0}^{t-1}\nu(k)\psi_i(k), \int_0^t \psi_i(\vartheta)\bar{\nu}(\vartheta)d\vartheta$ represent the present value respectively of an annuity-immediate, an annuity-due and a continuous annuity with variable rewards in the time and variable interest rate on a time horizon *t.*
The infinite rates cases are given by the limit to ∞ of the three relations. They converge depending on the values of ψ and ν .

$\nu(s,t) = \prod_{h=s+1}^{t}(1+r_h)^{-1}$, $\bar{\nu}(s,t) = e^{-\int_s^t \delta(\tau)d\tau}$ represent the $t-s$ *period discount factor at time s*, with homogeneous interest rates giving, at time *s,* the discounted value of one monetary unit to be paid at the end of period *k,* respectively in discrete and continuous time,

$\sum_{k=s+1}^{t}\nu(s,k)\psi_i(s,k), \sum_{k=s}^{t-1}\nu(s,k)\psi_i(s,k), \int_s^t \psi_i(s,\vartheta)\bar{\nu}(s,\vartheta)d\vartheta$ represent the present value at time *s* respectively of an annuity-immediate, an annuity-due and a continuous annuity paid on the time interval $(s,t]$ with non-homogeneous rewards and variable interest rate.

$\dot{\nu}(s,t) = \prod_{h=s+1}^{t}(1+{}_s r_h)^{-1}, \bar{\bar{\nu}}(s,\vartheta) = e^{-\int_s^\vartheta \delta(s,\tau)d\tau}$ represent the $t-s$ period discount factors at time *s*, with non-homogeneous interest rates,

$\sum_{k=s+1}^{t}\dot{\nu}(s,k)\psi_i(s,k), \sum_{k=s}^{t-1}\dot{\nu}(s,k)\psi_i(s,k), \int_s^t \psi_i(s,\vartheta)\bar{\bar{\nu}}(s,\vartheta)d\vartheta$ represent the present value at time *s* respectively of an annuity-immediate, an annuity-due and a continuous time annuity paid on the time interval $(s,t]$ with non-homogeneous rewards and non-homogeneous interest rate.
In the following sections, we will show that annuities are strongly related to the semi-Markov reward processes; for the Markov case see Janssen-Manca (2006).
A reward structure can be considered as a very general structure that, given a financial and economic meaning can be very useful in stochastic modelling.
For example, this behaviour is particularly efficient to construct models useful to follow the dynamic evolution of insurance problems. In this case, the

permanence in a state usually involves the payment of a premium or the receipt of a claim. Furthermore often the transition from one state to another induces some other cost or benefit.

5.2 Undiscounted SMRWP

For each given case we will present the immediate, the due and the continuous cases, both in homogeneous and non-homogeneous environments. We will give first the simplest case (only with permanence rewards and fixed rate of interest and rewards) and after the general ones. The same cases will be given for discounted processes.

5.2.1 Fixed Permanence Rewards

We assume that:
a) rewards are fixed in time,
b) rewards are given only for permanence in the state.
First we present the immediate case.
$V_i(t)$ $(V_i(s,t))$ represents the *Mean Total Reward* (MTR) paid or received in t periods (from time s to time t), given that at time 0 (at time s) the system was in state i.
At time 1 the evolution equation for the homogeneous immediate case is given by the following relation:

$$V_i(1) = (1-H_i(1))\psi_i + \sum_{k=1}^{m} b_{ik}(1)\psi_i + \sum_{k=1}^{m}\sum_{\vartheta=0}^{1} b_{ik}(\vartheta)V_k(1-\vartheta). \tag{5.1}$$

To have a good understanding of the evolution equation, let us first say that relations (1.8) and (1.11) imply that $b_{ij}(1)=Q_{ij}(1)$ and so relation (5.1) can be decomposed in the following way:

$$V_i(1) = \left(1-\sum_{k=1}^{m} Q_{ik}(1)\right)\psi_i + \sum_{k=1}^{m} Q_{ik}(1)\psi_i + \sum_{k=1}^{m}\sum_{\vartheta=0}^{1} Q_{ik}(\vartheta)V_k(1-\vartheta) \tag{5.2}$$

where $V_k(0)=0, \forall k$ and $Q_{ik}(0)=0\ \forall i,k$, and so:

$$V_i(1) = \psi_i. \tag{5.3}$$

For the next step, we can write that:

$$V_i(2) = (1-H_i(t))\psi_i 2 + \sum_{k=1}^{m}\sum_{\vartheta=1}^{2} b_{ik}(\vartheta)\psi_i\vartheta + \sum_{k=1}^{m}\sum_{\vartheta=1}^{2} b_{ik}(\vartheta)V_k(2-\vartheta). \tag{5.4}$$

This time, two rewards must be paid but in different ways. We divide the evolution equation in three parts.
- the term $(1-H_i(t))\psi_i 2$ represents the rewards obtained without state changes;

- the expression $\sum_{k=1}^{m}\sum_{\vartheta=1}^{2} b_{ik}(\vartheta)\psi_i\vartheta$ gives the rewards obtained before the change of state. As $b_{ik}(0)=0\ \forall i,k$, the sum on ϑ begins from 1;
- the double sum $\sum_{k=1}^{m}\sum_{\vartheta=1}^{2} b_{ik}(\vartheta)V_k(2-\vartheta)$ gives the rewards paid or earned after the transitions.

For time *t,* we get the following general result:

$$V_i(t) \ = \ (1-H_i(t))\psi_i t + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\psi_i\vartheta + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)V_k(t-\vartheta). \qquad (5.5)$$

The general formula in the non-homogeneous case is:

$$\begin{aligned} V_i(s,t) \ = \ & (1-H_i(s,t))(t-s)\psi_i + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)(\vartheta-s)\psi_i \\ & + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t). \end{aligned} \qquad (5.6)$$

In this simple case the due and the immediate processes correspond. So we report only the continuous cases.

$$V_i(t) = \left(1-H_i(t)\right)t\psi_i + \psi_i\sum_{k=1}^{m}\int_0^t \dot{Q}_{ik}(\vartheta)\vartheta d\vartheta + \sum_{k=1}^{m}\int_0^t \dot{Q}_{ik}(\vartheta)V_k(t-\vartheta)d\vartheta, \quad (5.7)$$

$$\begin{aligned} V_i(s,t) = & \left(1-H_i(s,t)\right)\cdot(t-s)\psi_i + \psi_i\sum_{k=1}^{m}\int_s^t \dot{Q}_{ik}(s,\vartheta)\cdot(\vartheta-s)d\vartheta \\ & + \sum_{k=1}^{m}\int_s^t \dot{Q}_{ik}(s,\vartheta)V_k(\vartheta,t)d\vartheta. \end{aligned} \qquad (5.8)$$

5.2.2 Variable Permanence And Transition Rewards

Here we assume that:

a) rewards are variable in time,

b) rewards are given for permanence in the state and at a given transition,

Under these hypotheses, we get respectively for homogeneous and non-homogeneous environments, in the immediate cases the following results:

$$\begin{aligned} V_i(t) \ = \ & (1-H_i(t))\sum_{\tau=1}^{t}\psi_i(\tau) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\sum_{\tau=1}^{\vartheta}\psi_i(\tau) \\ & + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\gamma_{ik}(\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)V_k(t-\vartheta), \end{aligned} \qquad (5.9)$$

$$\begin{aligned} V_i(s,t) &= (1-H_i(s,t))\sum_{\tau=s+1}^{t}\psi_i(\tau)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\sum_{\tau=1}^{\vartheta}\psi_i(\tau) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\gamma_{ik}(\vartheta)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)V_k(\vartheta,t). \end{aligned} \tag{5.10}$$

In the due case we obtain:

$$\begin{aligned} \ddot{V}_i(t) &= (1-H_i(t))\sum_{\tau=1}^{t}\psi_i(\tau-1)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\sum_{\tau=1}^{\vartheta}\psi_i(\tau-1) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\gamma_{ik}(\vartheta)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\ddot{V}_k(t-\vartheta), \end{aligned} \tag{5.11}$$

$$\begin{aligned} \ddot{V}_i(s,t) &= (1-H_i(s,t))\sum_{\tau=s+1}^{t}\psi_i(\tau-1)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\gamma_{ik}(\vartheta) \\ &+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\sum_{\tau=1}^{\vartheta}\psi_i(\tau-1)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t). \end{aligned} \tag{5.12}$$

The difference between immediate and due is given only by the time of payment of the rewards.

The continuous cases are the following:

$$\begin{aligned} V_i(t) &= \left(1-H_i(t)\right)\int_0^t\psi_i(\tau)d\tau+\sum_{k=1}^{m}\int_0^t\dot{Q}_{ik}(\vartheta)\int_0^{\vartheta}\psi_i(\tau)d\tau d\vartheta \\ &+\sum_{k=1}^{m}\int_0^t\dot{Q}_{ik}(\vartheta)\left(V_k(t-\vartheta)+\gamma_{ik}(\vartheta)\right)d\vartheta, \end{aligned} \tag{5.13}$$

$$\begin{aligned} V_i(s,t) &= \left(1-H_i(s,t)\right)\int_s^t\psi_i(\tau)d\tau+\sum_{k=1}^{m}\int_s^t\dot{Q}_{ik}(s,\vartheta)\int_s^{\vartheta}\psi_i(\tau)d\tau d\vartheta \\ &+\sum_{k=1}^{m}\int_s^t\dot{Q}_{ik}(s,\vartheta)\left(V_k(\vartheta,t)+\gamma_{ik}(\vartheta)\right)d\vartheta. \end{aligned} \tag{5.14}$$

The presence of the lump sums given or taken at the moment of transition times is taken into consideration.

5.2.3 Non-Homogeneous Permanence And Transition Rewards

In the last immediate case model, the rewards are non-homogeneous and so we have to consider only the non-homogeneous case.

Assumptions are thus:

a) rewards depend on the times *s* and *t*,

b) permanence and transition rewards are non-homogeneous.

Here, only the non-homogeneous case has sense and the evolution equations take the form:

$$
\begin{aligned}
V_i(s,t) &= (1-H_i(s,t))\sum_{\tau=s+1}^{t}\psi_i(s,\tau)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\sum_{\tau=s+1}^{\vartheta}\psi_i(s,\tau)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)V_k(\vartheta,t).
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
\ddot{V}_i(s,t) &= (1-H_i(s,t))\sum_{\tau=s+1}^{t}\psi_i(s,\tau-1)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\sum_{\tau=1}^{\vartheta}\psi_i(s,\tau-1)\\
&+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t).
\end{aligned}
\tag{5.16}
$$

$$
\begin{aligned}
V_i(s,t) &= \left(1-H_i(s,t)\right)\int_s^t\psi_i(s,\tau)d\tau+\sum_{k=1}^{m}\int_s^t\dot{Q}_{ik}(s,\vartheta)\int_s^{\vartheta}\psi_i(s,\tau)d\tau d\vartheta\\
&+\sum_{k=1}^{m}\int_s^t\dot{Q}_{ik}(s,\vartheta)\left(V_k(\vartheta,t)+\gamma_{ik}(s,\vartheta)\right)d\vartheta.
\end{aligned}
\tag{5.17}
$$

The other non-discounted cases can be treated in a similar way and are left to the reader, who can refer also to Janssen-Manca (2006).

5.3 Discounted SMRWP

For the discounted case developed in this section, we assume that all the rewards are discounted at time 0 in the homogeneous case and at time s in the non-homogeneous case. Let us point out that these models, as we will see below, are very important for insurance applications.

5.3.1. Fixed Permanence And Interest Rate Cases

In the first formulation of this case we suppose that:
a) rewards are fixed in time,
b) rewards are given only for permanence in the state,
c) interest rate r is fixed.
In this case $V_i(t)$ represents the *Rewards Mean Present Value* (RMPV) paid or received in a time t, given that at time 0 the system is in state i.
Under these hypotheses, a similar reasoning as before leads to the following result for the evolution equation, firstly for the homogeneous immediate case:

$$
V_i(1) = \left(1-H_i(1)\right)\psi_i\nu^1+\sum_{k=1}^{m}b_{ik}(1)\psi_i\nu^1+\sum_{k=1}^{m}\sum_{\vartheta=1}^{1}b_{ik}(\vartheta)V_k(1-\vartheta)\nu^1=\psi_i\nu^1, \tag{5.18}
$$

$$
V_i(t) = (1-H_i(t))\psi_i a_{\overline{t}|r}+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\psi_i a_{\overline{\vartheta}|r}+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)V_k(t-\vartheta)\nu^{\vartheta}. \tag{5.19}
$$

For the non-homogeneous case, this last result becomes:

$$\begin{aligned} V_i(s,t) &= (1-H_i(s,t))\psi_i a_{\overline{t-s}|r} + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\psi_i a_{\overline{\vartheta-s}|r} \\ &+ \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu^{\vartheta-s}. \end{aligned} \tag{5.20}$$

To explain these results, as for the continuous case, we divide the evolution equation in three parts. The meaning is the same given in the previous cases but we use annuity formulas.

Let us just make the following comments:

The term $(1-H_i(s,t))\psi_i a_{\overline{t-s}|r}$ represents the present value of the rewards obtained without state changes. More precisely $(1-H_i(s,t))$ is the probability of remaining in the state i and $\psi_i a_{\overline{t-s}|r}$ is the present value of a constant annuity of $t-s$ payments ψ_i.

The term $\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\psi_i a_{\overline{\vartheta-s}|r}$ gives the present value of the rewards t obtained before the change of state.

The term $\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu^{\vartheta-s}$ gives the present value of the rewards paid or earned after the transitions and as the change of state happens at time ϑ, it is necessary to discount the reward values at time s.

In the due environment we obtain:

$$\ddot{V}_i(t) = (1-H_i(t))\psi_i\ddot{a}_{\overline{t}|r} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\psi_i\ddot{a}_{\overline{\vartheta}|r} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\ddot{V}_k(t-\vartheta)\nu^{\vartheta-1}, \tag{5.21}$$

$$\begin{aligned} \ddot{V}_i(s,t) &= (1-H_i(s,t))\psi_i\ddot{a}_{\overline{t-s}|r} + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\psi_i\ddot{a}_{\overline{\vartheta-s}|r} \\ &+ \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t)\nu^{\vartheta-s-1}. \end{aligned} \tag{5.22}$$

At last the evolution equations in the continuous case are the following:

$$\begin{aligned} V_i(t) &= (1-H_i(t))\frac{1-e^{-\delta t}}{\delta}\psi_i + \sum_{k=1}^{m}\int_0^t \dot{Q}_{ik}(\vartheta)\frac{1-e^{-\delta\vartheta}}{\delta}\psi_i d\vartheta \\ &+ \sum_{k=1}^{m}\int_0^t \dot{Q}_{ik}(\vartheta)e^{-\delta\vartheta}V_k(t-\vartheta)d\vartheta, \end{aligned} \tag{5.23}$$

$$\begin{aligned} V_i(s,t) &= (1-H_i(s,t))\frac{1-e^{-\delta(t-s)}}{\delta}\psi_i + \sum_{k=1}^{m}\int_s^t \dot{Q}_{ik}(s,\vartheta)\frac{1-e^{-\delta(\vartheta-s)}}{\delta}\psi_i d\vartheta \\ &+ \sum_{k=1}^{m}\int_s^t \dot{Q}_{ik}(s,\vartheta)e^{-\delta(\vartheta-s)}V_k(\vartheta,t)d\vartheta. \end{aligned} \tag{5.24}$$

5.3.2 Variable Interest Rate, Permanence And Transition Cases

Now we make the following assumptions:
a) rewards are variable in time,
b) rewards are given for permanence in the state and at a given transition,
c) the interest rate is variable.
Under these hypotheses, in the immediate case we get the following relations:

$$\begin{aligned} V_i(t) \;=\; & (1-H_i(t))\sum_{h=1}^{t}\psi_i(h)\nu(h)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\sum_{h=1}^{\vartheta}\psi_i(h)\nu(h) \\ & +\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\gamma_{ik}(\vartheta)\nu(\vartheta)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)V_k(t-\vartheta)\nu(\vartheta), \end{aligned} \tag{5.25}$$

$$\begin{aligned} V_i(s,t) = & (1-H_i(s,t))\sum_{h=s+1}^{t}\psi_i(h)\nu(s,h)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\sum_{h=s+1}^{\vartheta}\psi_i(h)\nu(s,h) \\ & +\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\gamma_{ik}(\vartheta)\nu(s,\vartheta)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu(s,\vartheta). \end{aligned} \tag{5.26}$$

In the due case we get:

$$\begin{aligned} \ddot{V}_i(t) = & (1-H_i(t))\sum_{\tau=0}^{t-1}\psi_i(\tau)\nu(\tau)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{ik}(\vartheta)\sum_{\tau=0}^{\vartheta-1}\psi_i(\tau)\nu(\tau) \\ & +\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\nu(\vartheta-1)b_{ik}(\vartheta)\ddot{V}_k(t-\vartheta)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}\nu(\vartheta)b_{ik}(\vartheta)\gamma_{ik}(\vartheta), \end{aligned} \tag{5.27}$$

$$\begin{aligned} \ddot{V}_i(s,t) \;=\; & (1-H_i(s,t))\sum_{\tau=s}^{t-1}\psi_i(\tau)\nu(s,\tau)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}b_{ik}(s,\vartheta)\sum_{\tau=s}^{\vartheta-1}\psi_i(\tau)\nu(s,\tau) \\ & +\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\nu(s,\vartheta-1)b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t)+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t}\nu(s,\vartheta)b_{ik}(s,\vartheta)\gamma_{ik}(\vartheta). \end{aligned} \tag{5.28}$$

The evolution equations in the continuous case are the following:

$$\begin{aligned} V_i(t) = & \sum_{k=1}^{m}\int_0^t \dot{Q}_{ik}(\vartheta)e^{-\int_0^{\vartheta}\delta(\tau)d\tau}\left(V_k(t-\vartheta)+\gamma_{ik}(\vartheta)\right)d\vartheta \\ & +\left(1-H_i(t)\right)\int_0^t\psi_i(\theta)e^{-\int_0^{\theta}\delta(\tau)d\tau}d\theta+\sum_{k=1}^{m}\int_0^t\dot{Q}_{ik}(\vartheta)\int_0^{\vartheta}\psi_i(\theta)e^{-\int_0^{\theta}\delta(\tau)d\tau}d\theta d\vartheta, \end{aligned} \tag{5.29}$$

$$\begin{aligned} V_i(s,t) = & \sum_{k=1}^{m}\int_s^t \dot{Q}_{ik}(s,\vartheta)e^{-\int_s^{\vartheta}\delta(\tau)d\tau}\left(V_k(\vartheta,t)+\gamma_{ik}(\vartheta)\right)d\vartheta \\ & +\left(1-H_i(s,t)\right)\int_0^t\psi_i(\theta)e^{-\int_s^{\theta}\delta(\tau)d\tau}d\theta+\sum_{k=1}^{m}\int_s^t\dot{Q}_{ik}(s,\vartheta)\int_s^{\vartheta}\psi_i(\theta)e^{-\int_s^{\theta}\delta(\tau)d\tau}d\theta d\vartheta. \end{aligned} \tag{5.30}$$

5.3.3 Non-Homogeneous Interest Rate, Permanence And Transition Case

For our last case, we consider non-homogeneous rewards and interest rate. And so basic assumptions are:
a) rewards are non-homogeneous,
b) rewards are given for premanence and transitions,
c) interest rate is non-homogeneous.
It can easily be verified that the evolution equations take the form:

$$V_i(s,t) = (1 - H_i(s,t)) \sum_{\tau=s+1}^{t} \psi_i(s,\tau)\dot{v}(s,\tau) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) V_k(\vartheta,t)\dot{v}(s,\vartheta)$$
$$+\sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta)\dot{v}(s,\vartheta) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \sum_{\tau=s+1}^{\vartheta} \psi_i(s,\tau)\dot{v}(s,\tau), \tag{5.31}$$

$$\ddot{V}_i(s,t) = (1 - H_i(s,t))\sum_{\tau=s}^{t-1}\psi_i(s,\tau)\dot{v}(s,\tau) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} \dot{v}(s,\vartheta) b_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta)$$
$$+\sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} \dot{v}(s,\vartheta-1) b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\sum_{\tau=0}^{\vartheta-1}\psi_i(s,\tau)\dot{v}(s,\tau), \tag{5.32}$$

$$V_i(s,t) = \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) e^{-\int_s^{\vartheta} \delta(s,\tau)d\tau} \left(V_k(\vartheta,t) + \gamma_{ik}(s,\vartheta)\right) d\vartheta + \left(1 - H_i(s,t)\right)$$
$$\int_s^t \psi_i(s,\theta) e^{-\int_s^{\theta} \delta(s,\tau)d\tau} d\theta + \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) \int_s^{\vartheta} \psi_i(s,\theta) e^{-\int_s^{\theta} \delta(s,\tau)d\tau} d\theta d\vartheta. \tag{5.33}$$

6 GENERAL ALGORITHMS FOR DTSMRWP

In the previous section, we presented useful discrete time semi-Markov reward processes as well as general global models for which the evolution equations can be written in the matrix form

$$\mathbf{U} * \mathbf{V} = \mathbf{C}. \tag{6.1}$$

In the homogeneous case, **U** is an infinite order lower-triangular matrix whose elements are $m \times m$ matrices and **V** and **C** are infinite order vectors whose elements are m-dimensional vectors.
In the non-homogeneous case in (6.1) **U** is an infinite order upper-triangular matrix whose elements are $m \times m$ matrices and **V** and **C** are infinite order matrices whose elements are m-dimensional vectors.
Of course, matrices **U** and **C** depend on the particular models presented in the preceding section.
For real life applications, it is generally sufficient to study the problem on a finite time horizon $[0,T]$ and then the infinite system (6.1) becomes a finite system

$${}^{T}\mathbf{U} * {}^{T}\mathbf{V} = {}^{T}\mathbf{C} \tag{6.2}$$

where ${}^{T}\mathbf{U}$ is a square lower triangular block matrix of order $T+1$ in the homogeneous case and an upper triangular block matrix in the non-homogeneous case. ${}^{T}\mathbf{C}, {}^{T}\mathbf{V}$ are respectively $T+1$-dimensional vectors, in the homogeneous case, and matrices, in the non-homogeneous case, whose elements are *m*-dimensional vectors.
We will present briefly two general algorithms (homogenous and non-homogeneous) solving all possible reward cases.
The main steps of these algorithms are the following:

(i) *Homogeneous case*
Input – selectors that choose among the SMRWP, the number of states and the number of periods, the permanence and transition rewards, the fixed or variable interest rate, the transition matrix **P** and the matrix ${}^{T}\mathbf{F}$ of waiting time d.f.
Construct - ${}^{T}\mathbf{Q}$
Construct - ${}^{T}\mathbf{B}$
Construct - ${}^{T}\mathbf{H}$
Construct - ${}^{T}\mathbf{D}$
Construct – the permanence rewards
Construct – the transition rewards
Construct – the vector discount factors
Construct – ${}^{T}\mathbf{C}$, known terms
Solve - the system and find ${}^{T}\mathbf{V}$

(ii) *Non-homogeneous case*
Input – selectors, the number of states and the number of periods, the permanence and transition rewards, the fixed, variable or non-homogeneous interest rate, the transition matrix **P** and the matrix ${}^{T}\mathbf{F}$ waiting time d.f.

Construct - ${}^{T}\mathbf{Q}$
Construct - ${}^{T}\mathbf{B}$
Construct - ${}^{T}\mathbf{H}$
Construct - ${}^{T}\mathbf{D}$
Construct – the permanence rewards
Construct – the transition rewards
Construct – the matrix discount factors
Construct – ${}^{T}\mathbf{C}$, known terms
Solve - the system and find ${}^{T}\mathbf{V}$

These algorithms are able to solve any DTSMRWP. They constitute a very important tool for the application of semi-Markov reward processes in many applied sciences and, in this book, mainly in Finance, Insurance and Reliability.

7 NUMERICAL TREATMENT OF SMRWP

7.1 Undiscounted Case

Let us consider relations (5.7) and (5.13):

$$V_i(t) = \left(1 - H_i(t)\right)t\psi_i + \psi_i \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta)\vartheta d\vartheta + \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta) V_k(t-\vartheta) d\vartheta, \quad (7.1)$$

$$\begin{aligned} V_i(t) = {} & \left(1 - H_i(t)\right)\int_0^t \psi_i(\tau)d\tau + \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta) \int_0^\vartheta \psi_i(\tau) d\tau d\vartheta \\ & + \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta)\left(V_k(t-\vartheta) + \gamma_{ik}(\vartheta)\right) d\vartheta. \end{aligned} \quad (7.2)$$

For non-homogeneous models, the simplest and the most difficult cases are given by formulas (5.8) and (5.17), i.e:

$$\begin{aligned} V_i(s,t) = {} & \left(1 - H_i(s,t)\right) \cdot (t-s)\psi_i + \psi_i \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) \cdot (\vartheta - s) d\vartheta \\ & + \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) V_k(\vartheta,t) d\vartheta, \end{aligned} \quad (7.3)$$

$$\begin{aligned} V_i(s,t) = {} & \left(1 - H_i(s,t)\right)\int_s^t \psi_i(s,\tau)d\tau + \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) \int_s^\vartheta \psi_i(s,\tau) d\tau d\vartheta \\ & + \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta)\left(V_k(\vartheta,t) + \gamma_{ik}(s,\vartheta)\right) d\vartheta. \end{aligned} \quad (7.4)$$

Relations (7.2) and (7.4) can be written also as follows:

$$\begin{aligned} V_i(t) = {} & \left(1 - H_i(t)\right)\int_0^t \psi_i(\tau)d\tau + \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta) \int_0^\vartheta \psi_i(\tau) d\tau d\vartheta \\ & + \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta) V_k(t-\vartheta) d\vartheta + \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta)\gamma_{ik}(\vartheta) d\vartheta, \end{aligned} \quad (7.5)$$

$$\begin{aligned} V_i(s,t) = {} & \left(1 - H_i(s,t)\right)\int_s^t \psi_i(s,\tau)d\tau + \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) \int_s^\vartheta \psi_i(s,\tau) d\tau d\vartheta \\ & + \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) V_k(\vartheta,t) d\vartheta + \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta) d\vartheta. \end{aligned} \quad (7.6)$$

and so both the homogeneous and non-homogeneous integral equations can be written as follows:

$$V_i(t) = c_i(t) + \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta) V_k(t-\vartheta) d\vartheta, \tag{7.7}$$

$$V_i(s,t) = c_i(s,t) + \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) V_k(\vartheta,t) d\vartheta, \tag{7.8}$$

where for relations (7.1) and (7.3), we have:

$$c_i(t) = \left(1 - H_i(t)\right) t\psi_i + \psi_i \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta) \vartheta d\vartheta, \tag{7.9}$$

$$c_i(s,t) = \left(1 - H_i(s,t)\right) \cdot (t-s)\psi_i + \psi_i \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) \cdot (\vartheta - s) d\vartheta, \tag{7.10}$$

and for relations (7.2) (7.4):

$$\begin{aligned} c_i(t) = {} & \left(1 - H_i(t)\right) \int_0^t \psi_i(\tau) d\tau + \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta) \int_0^\vartheta \psi_i(\tau) d\tau d\vartheta \\ & + \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta) \gamma_{ik}(\vartheta) d\vartheta, \end{aligned} \tag{7.11}$$

$$\begin{aligned} c_i(s,t) = {} & \left(1 - H_i(s,t)\right) \int_s^t \psi_i(s,\tau) d\tau + \int_s^t \dot{Q}_{ik}(s,\vartheta) \gamma_{ik}(s,\vartheta) d\vartheta \\ & + \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) \int_s^\vartheta \psi_i(s,\tau) d\tau d\vartheta. \end{aligned} \tag{7.12}$$

As it can be seen, these last four equations have known terms that differ substantially but the coefficients of the two homogeneous integral equations are the same as in the non-homogeneous case.

Furthermore the integral equation (7.7) has the same coefficient as the equation (2.1) and the equation (7.8) the same as the equation (2.2).

As for the discretization of the homogeneous and non-homogeneous semi-Markov processes presented in section 2, we can consider the generic quadrature formula (2.3).

Let us recall that h is the step measure, $u,k \le N$, $u,k,N \in \mathbb{N}$, w, the weights related to the quadrature formula.

We also know that N is such that $Nh = Y$ and $\left[0, Y\right]$ is the integration interval.

Now, relations (7.7) and (7.8) can be discretized in the following way:

$$\tilde{V}_i(kh) = \tilde{c}_i(kh) + \sum_{l=1}^{m} \left(\sum_{\tau=0}^{k} w_{k\tau} \tilde{V}_l(kh - \tau h) \dot{Q}_{il}(\tau h) \right), \tag{7.13}$$

$$\tilde{V}_i(uh,kh) = \tilde{c}_{ij}(uh,kh) + \sum_{l=1}^{m} \left(\sum_{\tau=u}^{k} w_{u,k,\tau} \dot{Q}_{il}(uh,\tau h) \tilde{V}_i(\tau h, kh) \right), \tag{7.14}$$

but here, due to integral terms, the known term c should also be discretized.

Relations (7.13) and (7.14) in matrix form become:

$$\tilde{\mathbf{V}}(kh) = \tilde{\mathbf{c}}(kh) + \sum_{\tau=0}^{k} w_{k\tau} \dot{\mathbf{Q}}(\tau h) * \tilde{\mathbf{V}}(kh - \tau h), \tag{7.15}$$

$$\tilde{\mathbf{V}}(uh,kh) = \tilde{\mathbf{c}}(uh,kh) + \sum_{\tau=u}^{k} w_{uk\tau} \dot{\mathbf{Q}}(uh,\tau h) * \tilde{\mathbf{V}}(\tau h,kh), \tag{7.16}$$

or equivalently:

$$\tilde{\mathbf{V}}(kh) - w_{k0}\dot{\mathbf{Q}}(0) * \tilde{\mathbf{V}}(kh) = \tilde{\mathbf{f}}(kh) + \sum_{\tau=1}^{k} w_{k\tau} \dot{\mathbf{Q}}(\tau h) * \tilde{\mathbf{V}}(kh - \tau h), \tag{7.17}$$

$$\begin{aligned} &\tilde{\mathbf{V}}(uh,kh) - w_{uu\tau}\dot{\mathbf{Q}}(uh,uh) * \tilde{\mathbf{V}}(uh,kh) \\ &= \tilde{\mathbf{c}}(uh,kh) + \sum_{\tau=u+1}^{k} w_{uk\tau} \dot{\mathbf{Q}}(uh,\tau h) * \tilde{\mathbf{V}}(\tau h,kh). \end{aligned} \tag{7.18}$$

These last two relations take the form:

$$\left(\mathbf{I} - w_{k0}\dot{\mathbf{Q}}(0)\right) * \tilde{\mathbf{V}}(kh) = \tilde{\mathbf{f}}(kh) + \sum_{\tau=1}^{k} w_{k\tau} \dot{\mathbf{Q}}(\tau h) * \tilde{\mathbf{V}}(kh - \tau h), \tag{7.19}$$

$$\begin{aligned} &\left(\mathbf{I} - w_{uu\tau}\dot{\mathbf{Q}}(uh,uh)\right) * \tilde{\mathbf{V}}(uh,kh) \\ &= \tilde{\mathbf{c}}(uh,kh) + \sum_{\tau=u+1}^{k} w_{uk\tau} \dot{\mathbf{Q}}(uh,\tau h) * \tilde{\mathbf{V}}(\tau h,kh). \end{aligned} \tag{7.20}$$

Coefficient matrices of (7.19) and (7.20) are the same as those defined in systems (2.6) and (2.7) and so **Theorem 2.1** holds also in these cases.
The only difference is that the elements of the known terms are more difficult to construct.

7.2 Discounted Case

In this financial environment, we will consider the most complicated homogeneous and non-homogeneous discounted cases.
More precisely the CTHSMRWP related to relation (5.29) and the CTNHSMRWP formula (5.33) will be tackled.
As before we report the two relations:

$$\begin{aligned} V_i(t) &= \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta) e^{-\int_0^{\vartheta} \delta(\tau)d\tau} \left(V_k(t-\vartheta) + \gamma_{ik}(\vartheta) \right) d\vartheta \\ &+ \left(1 - H_i(t)\right) \int_0^t \psi_i(\theta) e^{-\int_0^{\theta} \delta(\tau)d\tau} d\theta + \sum_{k=1}^{m} \int_0^t \dot{Q}_{ik}(\vartheta) \int_0^{\vartheta} \psi_i(\theta) e^{-\int_0^{\theta} \delta(\tau)d\tau} d\theta d\vartheta, \end{aligned} \tag{7.21}$$

$$\begin{aligned} V_i(s,t) &= \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) e^{-\int_s^{\vartheta} \delta(s,\tau)d\tau} \left(V_k(\vartheta,t) + \gamma_{ik}(s,\vartheta) \right) d\vartheta + \left(1 - H_i(s,t)\right) \\ &\int_s^t \psi_i(s,\theta) e^{-\int_s^{\theta} \delta(s,\tau)d\tau} d\theta + \sum_{k=1}^{m} \int_s^t \dot{Q}_{ik}(s,\vartheta) \int_s^{\vartheta} \psi_i(s,\theta) e^{-\int_s^{\theta} \delta(s,\tau)d\tau} d\theta d\vartheta. \end{aligned} \tag{7.22}$$

Relations (7.21) and (7.22) can be written as follows:

$$\begin{aligned}V_i(t) &= \sum_{k=1}^m \int_0^t \dot{Q}_{ik}(\vartheta) e^{-\int_0^\vartheta \delta(\tau)d\tau} V_k(t-\vartheta)d\vartheta \sum_{k=1}^m \int_0^t \dot{Q}_{ik}(\vartheta) e^{-\int_0^\vartheta \delta(\tau)d\tau} \gamma_{ik}(\vartheta)d\vartheta \\ &+ (1-H_i(t)) \int_0^t \psi_i(\theta) e^{-\int_0^\theta \delta(\tau)d\tau} d\theta + \sum_{k=1}^m \int_0^t \dot{Q}_{ik}(\vartheta) \int_0^\vartheta \psi_i(\theta) e^{-\int_0^\theta \delta(\tau)d\tau} d\theta d\vartheta,\end{aligned} \tag{7.23}$$

$$\begin{aligned}V_i(s,t) &= \sum_{k=1}^m \int_s^t \dot{Q}_{ik}(s,\vartheta) e^{-\int_s^\vartheta \delta(s,\tau)d\tau} V_k(\vartheta,t)d\vartheta + \sum_{k=1}^m \int_s^t \dot{Q}_{ik}(s,\vartheta) e^{-\int_s^\vartheta \delta(s,\tau)d\tau} \gamma_{ik}(s,\vartheta)d\vartheta \\ &+ (1-H_i(s,t)) \int_s^t \psi_i(s,\theta) e^{-\int_s^\theta \delta(s,\tau)d\tau} d\theta + \sum_{k=1}^m \int_s^t \dot{Q}_{ik}(s,\vartheta) \int_s^\vartheta \psi_i(s,\theta) e^{-\int_s^\theta \delta(s,\tau)d\tau} d\theta d\vartheta.\end{aligned} \tag{7.24}$$

These equations take also the following form:

$$V_i(t) = \sum_{k=1}^m \int_0^t \dot{Q}_{ik}(\vartheta) e^{-\int_0^\vartheta \delta(\tau)d\tau} V_k(t-\vartheta)d\vartheta + c_i(t), \tag{7.25}$$

$$V_i(s,t) = \sum_{k=1}^m \int_s^t \dot{Q}_{ik}(s,\vartheta) e^{-\int_s^\vartheta \delta(s,\tau)d\tau} V_k(\vartheta,t)d\vartheta + c_i(s,t), \tag{7.26}$$

equations having the same structure.

This is also true for all the other cases with a fixed or variable intensity of interest rate.

We can thus proceed to the discretization procedure as before, to get the following relations:

$$\tilde{V}_i(kh) = \tilde{c}_i(kh) + \sum_{l=1}^m \left(\sum_{\tau=0}^k w_{k\tau} \tilde{V}_l(kh-\tau h) \dot{Q}_{il}(\tau h) \prod_{\theta=1}^\tau (1+r_h(\theta))^{-1} \right), \tag{7.27}$$

$$\begin{aligned}&\tilde{V}_i(uh,kh) = \tilde{c}_i(uh,kh) \\ &+ \sum_{l=1}^m \left(\sum_{\tau=u}^k w_{uk\tau} \tilde{V}_l(\tau h,kh) \dot{Q}_{il}(uh,\tau h) \prod_{\theta=u}^\tau (1+r_h(u,\theta))^{-1} \right).\end{aligned} \tag{7.28}$$

The functions $r_h(\theta)$ and $r_h(u,\theta)$ are the variable rates of interest obtained in this way:

$$\begin{aligned}(1+r_h(\theta)) &= \begin{cases} e^{\int_{(\theta-1)h}^{\theta h} \delta(\tau)d\tau}, & \theta \geq 1, \\ 1, & \theta = 0, \end{cases} \\ (1+r_h(u,\theta)) &= \begin{cases} e^{\int_{(\theta-1)h}^{\theta h} \delta(u,\tau)d\tau}, & \theta \geq u+1, \\ 1, & \theta = u. \end{cases}\end{aligned} \tag{7.29}$$

In matrix form, the equations can be written in the following way:

$$\tilde{\mathbf{V}}(kh) = \tilde{\mathbf{c}}(kh) + \sum_{\tau=0}^k w_{k\tau} \prod_{\theta=0}^\tau (1+r_h(\theta))^{-1} \dot{\mathbf{Q}}(\tau h) * \tilde{\mathbf{V}}(kh-\tau h), \tag{7.30}$$

$$\tilde{\mathbf{V}}(uh,kh)=\tilde{\mathbf{c}}(uh,kh)+\sum_{\tau=u}^{k}w_{uk\tau}\prod_{\theta=u}^{\tau}\left(1+r_h(u,\theta)\right)^{-1}\dot{\mathbf{Q}}(uh,\tau h)*\tilde{\mathbf{V}}(\tau h,kh) \quad (7.31)$$

or equivalently:

$$\begin{aligned}\tilde{\mathbf{V}}(kh)-w_{k0}\dot{\mathbf{Q}}(0)*\tilde{\mathbf{V}}(kh)&=\tilde{\mathbf{c}}(kh)\\&+\sum_{\tau=1}^{k}w_{k\tau}\prod_{\theta=0}^{\tau}\left(1+r_h(\theta)\right)^{-1}\dot{\mathbf{Q}}(\tau h)*\tilde{\mathbf{V}}(kh-\tau h),\end{aligned} \quad (7.32)$$

$$\begin{aligned}&\tilde{\mathbf{V}}(uh,kh)-w_{uu\tau}\dot{\mathbf{Q}}(uh,uh)*\tilde{\mathbf{V}}(uh,kh)\\&=\tilde{\mathbf{c}}(uh,kh)+\sum_{\tau=u+1}^{k}w_{uk\tau}\prod_{\theta=u}^{\tau}\left(1+r_h(u,\theta)\right)^{-1}\dot{\mathbf{Q}}(uh,\tau h)*\tilde{\mathbf{V}}(\tau h,kh).\end{aligned} \quad (7.33)$$

We can also write:

$$\begin{aligned}&\left(\mathbf{I}-w_{k0}\dot{\mathbf{Q}}(0)\right)*\tilde{\mathbf{V}}(kh)\\&=\tilde{\mathbf{c}}(kh)+\sum_{\tau=1}^{k}w_{k\tau}\prod_{\theta=0}^{\tau}\left(1+r_h(\theta)\right)^{-1}\dot{\mathbf{Q}}(\tau h)*\tilde{\mathbf{V}}(kh-\tau h),\end{aligned} \quad (7.34)$$

$$\begin{aligned}&\left(\mathbf{I}-w_{uu\tau}\dot{\mathbf{Q}}(uh,uh)\right)*\tilde{\mathbf{V}}(uh,kh)\\&=\tilde{\mathbf{c}}(uh,kh)+\sum_{\tau=u+1}^{k}w_{uk\tau}\prod_{\theta=u}^{\tau}\left(1+r_h(u,\theta)\right)^{-1}\dot{\mathbf{Q}}(uh,\tau h)*\tilde{\mathbf{V}}(\tau h,kh).\end{aligned} \quad (7.35)$$

So, we have shown that in the most difficult discounted cases, we get equations having the same coefficient matrices as the equations (7.19) and (7.20); consequently, here too, **Theorem 2.1** holds.

8. RELATION BETWEEN DTSMRWP AND SMRWP NUMERICAL SOLUTIONS

This section is related to the relation between the numerical solution of continuous and discrete time semi-Markov reward processes to show that, as in the SMP case, this approximation formula leads to the related discrete time process.

For simplicity, we just prove this result in the simplest numerical approach, i.e. the rectangle formula.

With this method, it is possible to evaluate the integral using the values of the function at the minimum of each of the discretization intervals or at the maximum

Clearly, in the continuous case to distinguish between the due and the immediate cases is meaningless and here we assume that reward payment times correspond to the “times” of the evaluations so that we could obtain respectively, by means of discretization, the due and the immediate cases. Here we only consider the immediate case.

8.1 Undiscounted Case

Taking into account relations (7.7) and (7.8), the general numerical solutions of a non-discounted CTHSMRWP and CTNHSMRWP can be written respectively in the following way:

$$\tilde{V}_i(kh) = c_i(kh) + h\sum_{\tau=0}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(\tau h)\tilde{V}_l(kh-\tau h), \tag{8.1}$$

$$\tilde{V}_i(uh,kh) = c_i(uh,kh) + h\sum_{\tau=u}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(uh,\tau h)\tilde{V}_l(\tau h,kh). \tag{8.2}$$

In the special case of relations (5.7) and (5.8), the equations (8.1) and (8.2) become:

$$\begin{aligned}\tilde{V}_i(kh) = &\left(1-H_i(kh)\right)k\psi_i + h\sum_{\tau=1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(\tau h)\tau\psi_i \\ &+ h\sum_{\tau=0}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(\tau h)\tilde{V}_l(kh-\tau h),\end{aligned} \tag{8.3}$$

$$\begin{aligned}\tilde{V}_i(uh,kh) = &\left(1-H_i(uh,kh)\right)(k-u)\psi_i + h\sum_{\tau=u+1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(uh,\tau h)(\tau-u)\psi_i \\ &+ h\sum_{\tau=u}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(uh,\tau h)\tilde{V}_l(\tau h,kh),\end{aligned} \tag{8.4}$$

where ψ_i represents the constant permanent reward paid at the end of each period.

Substituting the differential by means of the increment, we get:

$$\begin{aligned}\tilde{V}_i(kh) \cong &\left(1-H_i(kh)\right)k\psi_i + \sum_{\tau=1}^{k}\sum_{l=1}^{m}\left(Q_{il}(\tau h)-Q_{il}((\tau-1)h)\right)\tau\psi_i \\ &+ \sum_{\tau=0}^{k}\sum_{l=1}^{m}\left(Q_{il}(\tau h)-Q_{il}((\tau-1)h)\right)\tilde{V}_l(kh-\tau h),\end{aligned} \tag{8.5}$$

$$\begin{aligned}\tilde{V}_i(uh,kh) \cong &\left(1-H_i(uh,kh)\right)(k-u)\psi_i \\ &+ \sum_{\tau=u+1}^{k}\sum_{l=1}^{m}\left(Q_{il}(uh,\tau h)-Q_{il}(uh,(\tau-1)h)\right)(\tau-u)\psi_i \\ &+ \sum_{\tau=u}^{k}\sum_{l=1}^{m}\left(Q_{il}(uh,\tau h)-Q_{il}(uh,(\tau-1)h)\right)\tilde{V}_l(\tau h,kh).\end{aligned} \tag{8.6}$$

Using relations (1.11) and (1.12) and setting $h=1$, these results can be written in the form:

$$\tilde{V}_i(k) \cong \left(1-H_i(k)\right)k\psi_i + \sum_{\tau=1}^{k}\sum_{l=1}^{m}b_{il}(\tau)\tau\psi_i + \sum_{\tau=0}^{k}\sum_{l=1}^{m}b_{il}(\tau)\tilde{V}_l(k-\tau), \tag{8.7}$$

$$\tilde{V}_i(u,k) \cong (1-H_i(u,k))(k-u)\psi_i + \sum_{\tau=u+1}^{k}\sum_{l=1}^{m} b_{il}(u,\tau)(\tau-u)\psi_i$$
$$+\sum_{\tau=u}^{k}\sum_{l=1}^{m} b_{il}(u,\tau)\tilde{V}_l(\tau,k). \tag{8.8}$$

As we know that $b_{ij}(0)=0$ and $b_{ij}(u,u)=0$, $i,j=1,\ldots,m$, relations(8.7) and (8.8) correspond in fact to the evolution equations (5.5) and (5.6).

As for the second case, we will now give the numerical solution of the evolution equations of (5.13) and (5.17), respectively for homogeneous and non-homogeneous cases.

Let us recall these two equations:

$$\tilde{V}_i(kh) = (1-H_i(kh))h\sum_{\tau=1}^{k}\psi_i(\tau h) + h\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(\vartheta h)\gamma_{il}(\vartheta h)$$
$$+h\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\left(\dot{Q}_{il}(\vartheta h)h\sum_{\tau=1}^{\vartheta}\psi_i(\tau h)\right) + h\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(\vartheta h)\tilde{V}_l(kh-\vartheta h), \tag{8.9}$$

$$\tilde{V}_i(uh,kh) = (1-H_i(uh,kh))h\sum_{\tau=u+1}^{k}\psi_{il}(uh,\tau h)$$
$$+h\sum_{\vartheta=u+1}^{k}\sum_{l=1}^{m}\left(\dot{Q}_{il}(uh,\vartheta h)h\sum_{\tau=u+1}^{\vartheta}\psi_i(uh,\tau h)\right) \tag{8.10}$$
$$+h\sum_{\vartheta=u}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(uh,\vartheta h)\tilde{V}_l(\vartheta h,kh) + h\sum_{\vartheta=u}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(uh,\vartheta h)\gamma_{il}(uh,\vartheta h).$$

Proceeding as for the first case, we obtain:

$$\tilde{V}_i(k) = (1-H_i(k))\sum_{\tau=1}^{k}\psi_i(\tau) + \sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\left(b_{il}(\vartheta)\sum_{\tau=1}^{\vartheta}\psi_i(\tau)\right)$$
$$+\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}b_{il}(\vartheta)\gamma_{il}(\vartheta) + \sum_{\vartheta=1}^{k}\sum_{l=1}^{m}b_{il}(\vartheta)\tilde{V}_l(k-\vartheta), \tag{8.11}$$

$$\tilde{V}_i(u,k) = (1-H_i(u,k))\sum_{\tau=u+1}^{k}\psi_i(u,\tau) + \sum_{\vartheta=u}^{k}\sum_{l=1}^{m}b_{il}(u,\vartheta)\gamma_{il}(u,\vartheta)$$
$$+\sum_{\vartheta=u+1}^{k}\sum_{l=1}^{m}\left(b_{il}(u,\vartheta)\sum_{\tau=u+1}^{\vartheta}\psi_i(u,\tau)\right) + \sum_{\vartheta=u}^{k}\sum_{l=1}^{m}b_{il}(u,\vartheta)\tilde{V}_l(\vartheta,k), \tag{8.12}$$

so that relations (8.11) and (8.12) correspond in fact to relations (5.10) and (5.14).

8.2 Discounted Case

The change from continuous and discrete time in discounted cases implies that the financial discounting factors should change; in the case of constant intensity interest rate δ, it results that

$$\left(1+r_h\right)^{-1}=e^{-\delta h}, \tag{8.13}$$

and in the case of variable intensities respectively for homogeneous and non-homogeneous cases:

$$\begin{aligned} \left(1+r_h(\tau)\right)^{-1} &= e^{-\int_{h(\tau-1)}^{h\tau}\delta(\theta)d\theta}, \quad \tau=1,\dots,k, \\ \left(1+r_h(u,\tau)\right)^{-1} &= e^{-\int_{h(\tau-1)}^{h\tau}\delta(u,\theta)d\theta}, \quad \tau=u+1,\dots,k. \end{aligned} \tag{8.14}$$

(for more details on this topic see Volpe di Prignano (1985), Kellison(1991).
Now we will present two cases in the discounted environment: the discretization of relations (5.23), (5.24) and then of relations (5.29) and (5.33).
For the first case, the discretization method gives as relations:

$$\begin{aligned} \tilde{V}_i(kh) &= \left(1-H_i(kh)\right)a_{\overline{k}|r_h}\psi_i + h\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(\vartheta h)a_{\overline{\vartheta}|r_h}\psi_i \\ &+ h\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(\vartheta h)\tilde{V}_l(kh-\vartheta h)\left(1+r_h\right)^{-\vartheta}, \end{aligned} \tag{8.15}$$

$$\begin{aligned} \tilde{V}_i(uh,kh) &= \left(1-H_i(uh,kh)\right)a_{\overline{k-u}|r_h}\psi_i + h\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(uh,\vartheta h)a_{\overline{\vartheta-u}|r_h}\psi_i \\ &+ h\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(uh,\vartheta h)\tilde{V}_l(\vartheta h,kh)\left(1+r_h\right)^{-(\vartheta-u)}. \end{aligned} \tag{8.16}$$

Proceeding in the same way we use to get relations (8.7) and (8.8), the following results are obtained:

$$\begin{aligned} \tilde{V}_i(k) &= \left(1-H_i(k)\right)a_{\overline{k}|r}\psi_i + \sum_{\vartheta=1}^{k}\sum_{l=1}^{m}b_{il}(\vartheta)a_{\overline{\vartheta}|r}\psi_i \\ &+\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}b_{il}(\vartheta)\tilde{V}_l(k-\vartheta)\left(1+r\right)^{-\vartheta}, \end{aligned} \tag{8.17}$$

$$\begin{aligned} \tilde{V}_i(u,k) &= \left(1-H_i(u,k)\right)a_{\overline{k-u}|r}\psi_i + \sum_{\vartheta=1}^{k}\sum_{l=1}^{m}b_{il}(u,\vartheta)a_{\overline{\vartheta-u}|r}\psi_i \\ &+ \sum_{\vartheta=1}^{k}\sum_{l=1}^{m}b_{il}(u,\vartheta)\tilde{V}_l(\vartheta,k)\left(1+r\right)^{-(\vartheta-u)}, \end{aligned} \tag{8.18}$$

corresponding to relations (5.19) and (5.23).
For the second case, let us begin with the discretization of the equation (5.29) leading to:

$$\begin{aligned}\tilde{V}_i(kh) &= (1-H_i(kh))h\sum_{\vartheta=1}^{k}\psi_i(\vartheta h)\prod_{\tau=1}^{\vartheta}(1+r_h(\tau))^{-1} \\ &+ h\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(\vartheta h)h\sum_{\theta=1}^{\vartheta}\psi_i(\theta h)\prod_{\tau=1}^{\theta}(1+r_h(\tau))^{-1} \\ &+ h\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(\vartheta h)\gamma_{il}(\vartheta h)\prod_{\tau=1}^{\vartheta}(1+r_h(\tau))^{-1} \\ &+ h\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(\vartheta h)\tilde{V}_l(kh-\vartheta h)\prod_{\tau=1}^{\vartheta}(1+r_h(\tau))^{-1}.\end{aligned} \tag{8.19}$$

Once more, proceeding in the same way we use to find relations (8.7) and (8.8), we get:

$$\begin{aligned}\tilde{V}_i(k) &= \sum_{\vartheta=1}^{k}\sum_{l=1}^{m}b_{il}(\vartheta)\sum_{\theta=1}^{\vartheta}\psi_i(\theta)\prod_{\tau=1}^{\theta}(1+r(\tau))^{-1} \\ &+\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}b_{il}(\vartheta)\gamma_{il}(\vartheta)\prod_{\tau=1}^{\vartheta}(1+r(\tau))^{-1}+\sum_{\vartheta=1}^{k}\sum_{l=1}^{m}b_{il}(\vartheta)\tilde{V}_l(k-\vartheta)\prod_{\tau=1}^{\vartheta}(1+r(\tau))^{-1} \\ &+(1-H_i(k))\sum_{\vartheta=1}^{k}\psi_i(\vartheta)\prod_{\tau=1}^{\vartheta}(1+r(\tau))^{-1},\end{aligned} \tag{8.20}$$

this result corresponding to relation (5.25).
Finally, we have to dicretize the equation (5.33):

$$\begin{aligned}\tilde{V}_i(uh,kh) &= h\sum_{\vartheta=u+1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(uh,\vartheta h)h\sum_{\theta=u+1}^{\vartheta}\psi_i(uh,\theta h)\prod_{\tau=1}^{\theta}(1+r_h(u,\tau))^{-1} \\ &+(1-H_i(uh,kh))h\sum_{\vartheta=u+1}^{k}\psi_i(uh,\vartheta h)\prod_{\tau=u+1}^{\vartheta}(1+r_h(u,\tau))^{-1} \\ &+h\sum_{\vartheta=u+1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(uh,\vartheta h)\gamma_{il}(uh,\vartheta h)\prod_{\tau=u+1}^{\vartheta}(1+r_h(u,\tau))^{-1} \\ &+h\sum_{\vartheta=u+1}^{k}\sum_{l=1}^{m}\dot{Q}_{il}(uh,\vartheta h)\tilde{V}_l(\vartheta h,kh)\prod_{\tau=u+1}^{\vartheta}(1+r_h(u,\tau))^{-1}.\end{aligned} \tag{8.21}$$

And as above, we get the result corresponding to relation (5.31)

$$\begin{aligned}\tilde{V}_i(u,k) &= \sum_{\vartheta=u+1}^{k}\sum_{l=1}^{m}b_{il}(u,\vartheta)\sum_{\theta=u+1}^{\vartheta}\psi_i(u,\theta)\prod_{\tau=1}^{\theta}(1+r(u,\tau))^{-1} \\ &+(1-H_i(u,k))\sum_{\vartheta=u+1}^{k}\psi_{il}(u,\vartheta)\prod_{\tau=u+1}^{\vartheta}(1+r(u,\tau))^{-1} \\ &+\sum_{\vartheta=u+1}^{k}\sum_{l=1}^{m}b_{il}(u,\vartheta)\gamma_{il}(u,\vartheta)\prod_{\tau=u+1}^{\vartheta}(1+r(u,\tau))^{-1} \\ &+\sum_{\vartheta=u+1}^{k}\sum_{l=1}^{m}b_{il}(u,\vartheta)\tilde{V}_l(\vartheta,k)\prod_{\tau=u+1}^{\vartheta}(1+r(u,\tau))^{-1}.\end{aligned} \tag{8.22}$$

Chapter 5

SEMI-MARKOV EXTENSIONS OF THE BLACK-SCHOLES MODEL

1 INTRODUCTION TO OPTION THEORY

During the last thirty years, financial innovation has generalised the systematic use of new financial instruments such as *options* and *swaps*, mainly motivated for hedging but also, sometimes, used as speculative tools.
So, let us begin by recalling the basic definition of *option theory*.

Definition 1.1 *A* **call option** (*resp.* **put option)** *is a contract giving the right to buy* (*resp. to sell*) *a financial asset*, *called an underlying asset, for a fixed price*, *called exercise price*, *at the end of the contract time, called maturity time, also laid down in the contract*.

If one can exercise the option at any time before maturity, this type of option is called of an *American type*; if one can exercise it only at maturity, the option is called of a *European type*.
Let us use the following notation:
K: exercise price,
T: maturity time,
S: value of the underlying asset at maturity.
Then the "gain" of the holder of a European option at maturity time *T* is represented by the following graph.

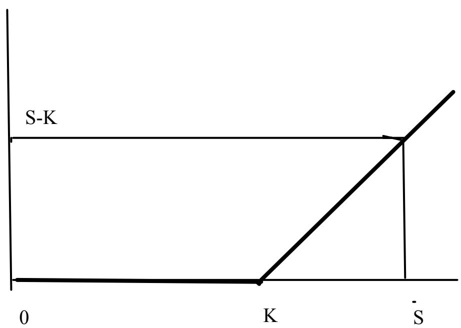

Figure 1.1: call option: holder's gain at maturity

For the holder of a put, this graph becomes:

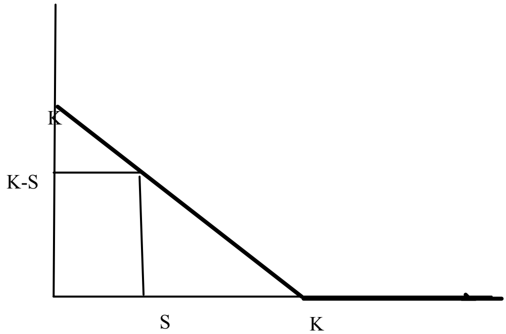

Figure 1.2: put option: holder's gain at maturity

Of course, to get the "net gain", we must estimate the cost of the option, often called *option premium*, and furthermore transaction costs and taxes.
Let us represent respectively by C and P the premiums of call and put options.
So, we get, without taking into account transaction costs and taxes, the following two graphical representations

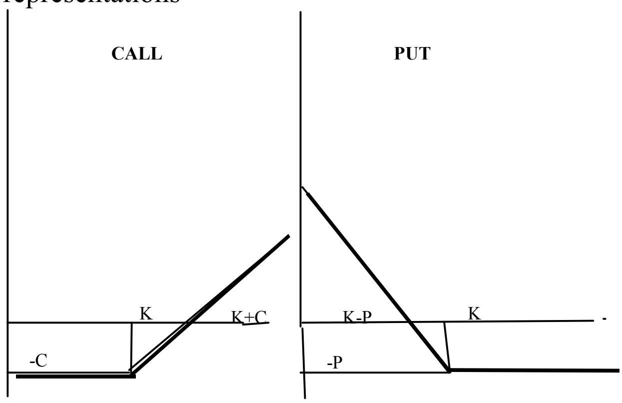

Figure 1.3: call and put options: net gains at maturity for the holder

The main problem is called the *pricing of optional products,* that is to give within the framework of an economic-financial theory framework, the values of premiums C and P as a function of the maturity T and the value of the asset at time 0.
More generally, as the holder of an option can sell his option on the option market at any time t, $0<t<T$, it is also necessary to give the "fair" value of the option at this time t knowing that the underlying asset has, at this time, the value $S=S(t)$.

Commonly, this fair market value is represented by

$$C(S,\tau) \tag{1.1}$$

where

$$\tau = T - t \tag{1.2}$$

represents the maturity computed at time *t*.
Sometimes, it is also useful to represent the call value as a function of the time *C*(*S,t*).
We see here that it is absolutely necessary to make assumptions about the stochastic process

$$S = \left(S(t), 0 \le t \le T\right). \tag{1.3}$$

Concerning the economic-financial theory framework, we adopt the assumption of *efficient market,* meaning that all the information available at time *t* is reflected in the values of the assets and so, transactions having an abnormally high profitability are not possible.
More precisely, an efficient market satisfies the following assumptions:
1. absence of transaction costs,
2. possibility of short sales,
3. availability of all information to all the economic agents,
4. perfect divisibility of assets,
6. continuous time financial market.

Furthermore, the market is *complete*; meaning that there exist zero-coupon bonds without risk for all possible maturities.
Let us remark that the word "information" used in point 3 can have different interpretations: weak, semi-strong or strong depending on if it is based on past prices, or on all public information or finally on all possible information that the agent can find.
According to Fama (1965), the efficient assumption justifies the "random walk" model in discrete time, saying that if $\Delta R_i(s)$ represents the increment of an asset *i* between *s* and *s*+1, we have:

$$\Delta R_i(s) = \mu_i + \varepsilon_i(s), \tag{1.4}$$

μ_i being a constant and $(\varepsilon_i(s))$ a sequence of uncorrelated r.v. of mean 0, sometimes called *errors*.
If we add the assumptions of equality of variances and of normality of the sequence $(\varepsilon_i(s))$, we get in fact a special case of the classical random walk introduced in Chapter 3.
If the efficiency assumption seems to be natural, some empirical studies show that it is not always the case in particular, since some agents can have access to preference information in principle forbidden by the legal authority.
Nevertheless, should such agents use the pertinent information it will be quickly noticeable by those markets and balance between agents will be restored.
This possibility, also called the case of *asymmetric information,* was studied by

Spence, Akerlof and Stiglitz, who was awarded the Nobel Prize in Economics in 2001.
From a personal point of view, the authors think that if the efficiency assumption seems quite normal for the long term, i.e., with a time unit large enough, however, it does not always seem to be true locally, i.e., with a short time unit. Indeed deregulation of markets where investors are willing to accept very small benefits in a short time but with many transactions plainly explains the intense activity of, for example, the currency markets which get very small benefits.
Due to the possibility of arbitrage, this is virtually making money without any investment otherwise known as "free lunch".
That is why models for *asymmetric information* should always be short term models rejecting the AOA assumption.
To be complete, let us remark that it is now possible to construct models without the AOA assumption but with assumptions on the time asset evolution and a selection of different possible scenarios, so that the investor can predict what will happen if such scenarios occur (cf Janssen, Manca et Di Biase (1997) and Jousseaume (1995)).
To conclude this section, let us emphasize the fact that traditional option pricing needs the efficiency of market dynamics and so of the AOA and also the choice of a stochastic model for the underlying asset time evolution.
Therefore we will begin this chapter with a presentation of the two most used classical models: the **Cox-Ross-Rubinstein** model in discrete time and the **Black-Scholes** model in continuous time. Then we will give the semi-Markov extension of these two models presented by Janssen and Manca (1999) and finally a non semi-Markov model with possibility of arbitrage (Janssen, Manca et Di Biase (1997)).

2 THE COX-ROSS-RUBINSTEIN (CRR) OR BINOMIAL MODEL

The model we will present here has the advantage of being quite simple in a financial world not always open to the use of sophisticated mathematical tools such as those used by Black and Scholes in 1973 to get their famous formula. And so the CRR model, though coming later, was very good for the use of the BS formula since, in the limit, the CRR model gives this formula again.
Moreover, the CRR model has still its own utility for financial institutions using discrete time models even with a short time period.

2.1 One-Period Model

To begin with, let us consider a model with only one time period, from time 0 to time 1; the time unit can be chosen as the user wishes: a quarter, a month, a week, a day, an hour,….

The basic assumption concerning the stochastic evolution of the underlying asset is that, starting from value $S(0) = S_0$ at time 0, it can only get two values at the end of the time period: $uS_0 (u > 1)$ if there is an up movement or $dS_0 (0 < d < 1)$ in the case of a down movement, parameter u and d being supposed to be known for the moment.

The probability measure is thus defined by the probability q of an up movement and to avoid trivialities, we will assume that:

$$0 < q < 1. \tag{2.1}$$

The next figure shows the two possible trajectories with of course

$$p = 1 - q. \tag{2.2}$$

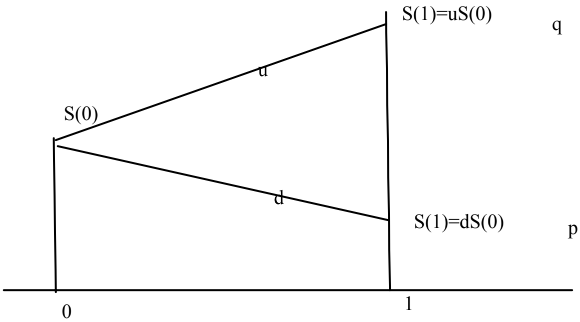

Figure 2.1: one-period binomial model

If one prefers to work with the percentages x and y respectively of gain and loss, we can express u and d as follows:

$$u = \left(1 + \frac{x}{100}\right), d = \left(1 - \frac{y}{100}\right). \tag{2.3}$$

We also suppose that there is no dividend distribution during the period.

Let us now consider an investor wishing to buy a European call on time 0 with maturity 1 and with K as exercise price.

The **problem** is thus to fix the *premium* of this call, that which the investor has to pay at time 0 to buy this call, knowing the value S_0 of the underlying asset at time 0.

2.1.1 The Arbitrage Model

If the investor wants to buy a call, it is clear that he anticipates an up movement of the call so that exercising the call at the end of the period will be advantageous for him and of course for the seller of the call the reverse will happen.
Nevertheless, the investor would take as little risk as possible knowing that he has always the possibility to invest on the non-risky market giving a fixed interest rate i per period.
To build a theory taking into account the apparently contradictory points of view, modern financial theory is based on the principle of *absence of arbitrage opportunity* (in short the AOA principle) meaning that there is no possibility to gain money without any investment, that is, there is no possibility of getting a *free lunch*.
This principle implies that the parameters d, u and i of the model must satisfy the following inequalities:

$$d < 1 + i < u. \tag{2.4}$$

Indeed, let us suppose for example that the first inequality is not true. In this case the investment in the asset is always better than the investment in the non-risky market. Then if we borrow the sum S_0 from the bank and buy the asset, at the end of the period we gain for sure the amount $\left(d-(1+i)\right)S_0$, and this is a free lunch.
Similarly, if the right-hand inequality is false, we can sell the asset at time 0 to get it to the seller at time 1 and so, the minimum value of the free lunch is in this case $(1+i-u)S_0$, *so that in both cases,* the AOA principle is not satisfied.
As the seller of a call option, for example, has the obligation to sell the shares if the holder of the call exercises his right, he must be able to do it whatever the value of the considered share is; that is why we have to introduce the important concept of *hedging*.
To do so, let us consider a portfolio in which at time 0 we have Δ shares and an amount B of money invested at the non-risky rate i per period.
B may be negative in case of a loan given by the bank.
Under the AOA assumption, the investment in the call must follow the same random evolution as the considered portfolio so that we have the following relations for t=1:

$$\begin{aligned} C_u(1) &= uS_0 + (1+i)B_0, \\ C_d(1) &= dS_0 + (1+i)B_0, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} C_u(1) &= \max\{0, uS_0 - K\}, \\ C_d(1) &= \max\{0, dS_0 - K\}. \end{aligned} \tag{2.6}$$

The system (2.5) is a linear system with two unknown values Δ, B.
The unique solution is given by:

$$\Delta = \frac{C_u(1) - C_d(1)}{(u-d)S_0},$$
$$B = \frac{uC_d(1) - dC_u(1)}{(u-d)(1+i)}. \tag{2.7}$$

Now, as said above, from the AOA assumption, the value of the call at $t = 0$, denoted for the moment by $C(S_0, 0)$, is equal to the initial value of the portfolio so that:

$$C(S_0,1) = S_0\Delta + B_0,$$
$$C(S_0,1) = S_0 \frac{C_u(1) - C_d(1)}{(u-d)S_0} + \frac{uC_d(1) - dC_u(1)}{(u-d)(1+i)}. \tag{2.8}$$

We can also write this value in the following form:

$$C(S_0,0) = \frac{1}{1+i}\left[\overline{q}C_u(1) + (1-\overline{q})C_d(1)\right],$$
$$\overline{q} = \frac{1+i-d}{u-d}. \tag{2.9}$$

This last expression shows that the value of the call at the beginning of the period can be seen as the *present value of the expected value of the "gain" at the end of the period.* But this expectation is computed under a new probability measure defined by $\overline{q}$, called *risk neutral measure* in opposition to the initial measure defined by q, and called the *historical* or *physical measure*.

From assumption (2.4), this risk neutral measure is *unique* and moreover independent of q, that is on the historical measure.

This shows that whatever the investor anticipates about the price of the considered underlying asset, using this model, he will always get the same result as another investor.

However, it must be clear that this risk neutral measure only gives an easy way to compute the "fair" value of the call, but if we want to compute probabilities of events, such as for example the probability of exercising the call at the end of the period, then it is the historical measure that must be used.

2.1.2 Numerical Example

Let us consider the data

$$S_0 = 80, K = 80, u = 1.5, d = 0.5, i = 3\%. \tag{2.10}$$

It follows from the following model:

$$C_u(1) = \max\left\{0.80 \times 1.5 - 80\right\} = 40,$$
$$C_d(1) = \max\left\{0.80 \times 0.5 - 80\right\} = 0. \tag{2.11}$$

The value of $\overline{q}$ is obtained, i.e.,

$$\overline{q} = \frac{1.03 - 0.5}{1.5 - 0.5} = 0.53 \qquad (2.12)$$

and so we get the option value

$$C_{fin}(80,0) = \frac{1}{1.03}\left[\overline{q} \times 40 + (1-\overline{q}) \times 0\right] = 20.5825. \qquad (2.13)$$

2.2 Multi-Period Model

2.2.1 Case Of Two Periods

The two following figures show how the model with two periods works.
Here we have to evaluate not only the value of the call at the origin but also at the intermediary time t=1.

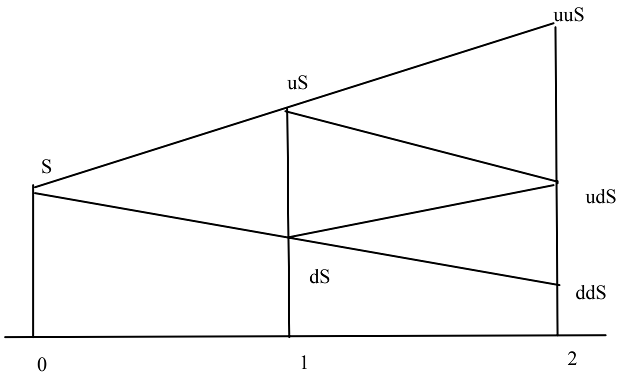

Figure 2.2: two-period model: scenarios for the underlying asset

Using the notation $C(S,t), t = 0,1,2$ in which the second variable represents the time, here 0, 1 or 2, the first one is the value of the underlying asset at this considered time.
Here too, as in the case of only one period, the call values will be assessed with the risk neutral measure as the present values at time t of the "gains" at maturity $t=2$ i.e.:

$$E_{\overline{q}}\left(C(S,2)\right). \qquad (2.14)$$

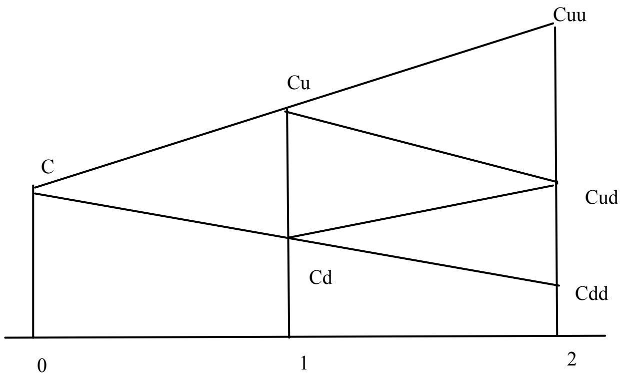

Figure.2.3: two-period model: values of the call

For example we get for $t = 0$:

$$C(S_0,0) = \frac{1}{(1+i)^2}\begin{bmatrix} \overline{q}^2 \max\left\{0, u^2 S_0 - K\right\} + 2\overline{q}(1-\overline{q}) \\ \cdot \max\left\{0, udS_0 - K\right\} + (1\text{-}\overline{q})^2 \max\left\{0, d^2 S_0 - K\right\} \end{bmatrix}. \quad (2.15)$$

Remark 2.1 Using a backward reasoning from t=2 to t=1 and from t=1 to t=0, it is also possible to get this last result since in fact:

$$\begin{aligned} C(uS_0,1) &= \frac{1}{1+i}\left[\overline{q}C\ (u^2 S_0,2) + (1-\overline{q})C(udS_0,2)\right], \\ C(dS_0,1) &= \frac{1}{1+i}\left[\overline{q}C(udS_0,2) + (1-\overline{q})C(d^2 S_0,2)\right], \\ C(S_0,0) &= \frac{1}{1+i}\left[\overline{q}C(uS_0,1) + (1-\overline{q})C(dS_0,1)\right]. \end{aligned} \quad (2.16)$$

Substituting the first two values in the last equality given just above, we get back to relation (2.15).

2.2.2 Case Of n Periods

If $C_{u^j d^{n-j}}(S_0,n)$ represents the call value at t=n if the underlying asset has had j up movements and $n-j$ down movements and with an initial value of the underlying asset of $S(0)$, that is:

$$C_{u^j d^{n-j}}(S_0,n) = \max\left\{0, u^j d^{n-j} S_0 - K\right\}, j = 0,1,...,n, \quad (2.17)$$

a straightforward extension of the case of two periods gives the following result:

$$C(S_0,0) = \frac{1}{(1+i)^n}\sum_{j=0}^{n}\binom{n}{j}\overline{q}^j (1-\overline{q})^{n-j} C_{u^j d^{n-j}}(n) \quad (2.18)$$

and similar results for intermediary time values.
From the computational point of view, Cox and Rubinstein introduced the minimum number of up movements a so that the call will be "in the money", that will mean that the holder has the advantage to exercise his option; clearly, this integer is given by:

$$a = \min\left\{ j \in N : u^j d^{n-j} S_0 > K \right\}. \tag{2.19}$$

Of course, if a is strictly larger than n, the call will always finish "out of the money" so that the call value at $t=n$ is null.
From relation (2.19), we get:

$$u^j d^{n-j} S_0 = K \Leftrightarrow a = \left\lfloor \frac{\log K S_0^{-1} d^{-n}}{\log u d^{-1}} \right\rfloor + 1, \tag{2.20}$$

$\lfloor x \rfloor$ representing the larger integer smaller than or equal to the real x.
From Chapter 1, section 5.1, we know that if X is a r.v. having a binomial distribution with parameters (n,q), we have:

$$P\left(X > a-1\right) = \sum_{j=a}^{n} \binom{n}{j} q^j (1-q)^{n-j} (= \bar{B}(n,q;a)). \tag{2.21}$$

As we have (see Cox, Rubinstein (1985), p.178):

$$\bar{q} < \frac{1+i}{u} < 1, \tag{2.22}$$

it follows that the quantity $\bar{q}'$ defined below is such that $0 < \bar{q}' < 1$ and so the call value can be written in the form:

$$\begin{aligned} C_{fin}(S_0,0) &= S_0 \bar{B}(n,\bar{q}';a) - \frac{K}{(1+i)^n} \bar{B}(n,\bar{q};a), \\ \bar{q} &= \frac{1+i-d}{u-d}, \bar{q}' = \frac{u}{1+i} q. \end{aligned} \tag{2.23}$$

In conclusion, the binomial distribution is sufficient to compute the call values.

2.2.3 Numerical Example

Coming back to the preceding example for which

$$S_0 = 80, K = 80, u = 1.5, d = 0.5,\ i = 3\%, \tag{2.24}$$

and $\bar{q} = 0.53$ but now for $n=2$, we get:

$$\bar{q}' = \frac{1.5}{1.03} \times 0.6 = 0.7718 \tag{2.25}$$

and consequently

$$C(80,0) = 26.4775. \tag{2.26}$$

3 THE BLACK-SCHOLES FORMULA AS LIMIT OF THE BINOMIAL MODEL

3.1 The Log-Normality Of The Underlying Asset

Since nowadays financial markets operate in continuous time, we will study the asymptotic behaviour of the CRR formula (2.23) to obtain the value of a call at time 0 and of maturity T.

To begin with, we will work with a discrete time scale on $[0,T]$ with a unit time period h defined by $n=T/h$ or more precisely $n=\lfloor T/h \rfloor$.

Moreover, if i represents the annual interest rate, the rate for a period of length h called $\hat{i}$ is defined by the relation:

$$(1+\hat{i})^n = (1+i)^T , \tag{3.1}$$

so that

$$\hat{i} = (1+i)^{T/n} - 1. \tag{3.2}$$

If J_n represents the r.v. giving the number of ascending movements of the underlying asset, we know that:

$$J_n \prec B(n,q) \tag{3.3}$$

and so, starting from S_0, the value of the underlying asset at the end of period n is given by

$$S(n) = u^{J_{,n}} d^{n-J_{,n}} S_0. \tag{3.4}$$

It follows that

$$\log \frac{S(n)}{S_0} = J_n \log \frac{u}{d} + n \log d. \tag{3.5}$$

The results of the binomial distribution (see Chapter 1, section 5.1) imply that

$$\begin{aligned} &E\left(\log \frac{S(n)}{S_0}\right) = \hat{\mu} n, \\ &\operatorname{var}\left(\log \frac{S(n)}{S_0}\right) = \hat{\sigma}^2 n, \\ &\hat{\mu} = q\hat{\sigma}^2 + \log d, \\ &\hat{\sigma}^2 = q(1-q)\left(\log \frac{u}{d}\right)^2. \end{aligned} \tag{3.6}$$

To obtain a limit behaviour, for every fixed n, we must introduce a dependence of u, d and q with respect to $n=\lfloor T/h \rfloor$ so that

$$\lim_{n\to\infty} \hat{\mu}(n)n = \alpha T,$$
$$\lim_{n\to\infty} \hat{\sigma}^2(n)n = \sigma^2 T, \tag{3.7}$$

α,σ being constant values as parameters of the limit model. As an example, Cox and Rubinstein (1985) select the values

$$u = e^{\sigma\sqrt{T/n}}, d = \frac{1}{u}(= e^{-\sigma\sqrt{T/n}}),$$
$$q = \frac{1}{2} + \frac{1}{2}\frac{\alpha}{\sigma}\sqrt{T/n}. \tag{3.8}$$

This choice leads to the values:

$$\hat{\mu}(n)n = \alpha T,$$
$$\hat{\sigma}^2(n)n = \left(\sigma^2 - \alpha^2 \frac{T}{n}\right)T. \tag{3.9}$$

Using a version of the central limit theorem for independent but non-identically distributed r.v., the authors show that $S(n)/S_0$ converges in law to a log-normal distribution for $n \to \infty$. More precisely, we have:

$$P\left(\frac{\log\frac{S(n)}{S_0} - \hat{\mu}(n)n}{\hat{\sigma}\sqrt{n}} \leq x\right) \to \Phi(x), \tag{3.10}$$

Φ being as defined in Chapter 1, section 5.3, the distribution function of the reduced normal distribution provided that the following condition is satisfied:

$$\frac{q\left|\log u - \hat{\mu}\right|^3 + (1-q)\left|\log u - \hat{\mu}\right|^3}{\hat{\sigma}^3\sqrt{n}} \xrightarrow[n\to\infty]{} 0. \tag{3.11}$$

This condition is equivalent to

$$\frac{(1-q)^2 + q^2}{\sqrt{nq(1-q)}} \to 0 \tag{3.12}$$

which is true from assumption (3.8).
This result and the definition given in Chapter 1, section 5.4, give the next proposition:

Proposition 3.1(*Cox and Rubinstein* (*1985*))
Under the assumptions (3.8)*, the limit law of the underlying asset is a lognormal law with parameters* $(\alpha T, \sigma^2 T)$ *or*

$$P\left(\frac{\log\frac{S(T)}{S_0} - \alpha T}{\sigma\sqrt{T}} \leq x\right) = \Phi(x). \tag{3.13}$$

In particular, it follows that:

$$E\left(\frac{S(T)}{S_0}\right)=e^{\alpha T+\frac{\sigma^2}{2}T},$$
$$\operatorname{var}\left(\frac{S(T)}{S_0}\right)=e^{2\alpha T+\sigma^2 T}(e^{\sigma^2 T}-1). \tag{3.14}$$

3.2 The Black-Scholes Formula

Starting from the result (2.21) and using **Proposition 3.1** under the risk neutral measure, Cox & Rubinstein (1985) proved that the asymptotic value of the call is given by the famous **Black and Scholes** (1973) formula:

$$C(S,T)=S\Phi(x)-K(1+i)^{-T}\Phi(x-\sigma\sqrt{T}),$$
$$x=\frac{\ln\left(S/K(1+i)^{-T}\right)}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T}. \tag{3.15}$$

Here, we note the call using the maturity as second variable and S representing the value of the underlying asset at time 0.

The interpretation of the Black and Scholes formula can be given with the concept of a hedging portfolio.

Indeed, we already know that in the CRR model, the value of the call takes the form:

$$C=S\Delta+B, \tag{3.16}$$

Δ representing the proportion of assets in the portfolio and B the quantity invested on the non-risky rate at t=0.

From the result (3.16), at the limit, we obtain:

$$\Delta=\Phi(x),$$
$$B=-K(1+i)^{-T}\Phi(x-\sigma\sqrt{T}). \tag{3.17}$$

So, under the assumption of an efficient market, the hedging portfolio is also known in continuous time.

Remark 3.1 This hedging portfolio must of course, at least theoretically, be rebalanced at every time s on $[0,T]$. Rewriting the Black and Scholes formula for computing the call at time s, the underlying asset having the value S, we get:

$$\Delta=\Phi(x), B=-K(1+i)^{-(T-s)}\Phi(x-\sigma\sqrt{T-s}),$$
$$x=\frac{\ln\left(S/K(1+i)^{-(T-s)}\right)}{\sigma\sqrt{T-s}}+\frac{1}{2}\sigma\sqrt{T-s}. \tag{3.18}$$

Of course a continuous rebalancing and even a portfolio with frequent time changes are not possible due to the transaction costs.

4 THE BLACK-SCHOLES CONTINUOUS TIME MODEL

4.1 The Model

In fact, Black and Scholes used a continuous time model for the underlying asset introduced by Samuelson (1965).
On a complete filtered probability space $\left(\Omega, \Im, \left(\Im_t, t \geq 0\right), P\right)$ (see **Definition 7.2** of Chapter 1) the stochastic process

$$S = \left(S(t), t \geq 0\right) \tag{4.1}$$

will now represent the time evolution of the underlying asset.
The basic assumption is that the stochastic dynamic of the *S*-process is given by

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dB(t), \\ S(0) &= S_0, \end{aligned} \tag{4.2}$$

where the process $B = (B(t), t \in [0,T])$ is a standard Brownian process (see Chapter 1, section 9 adapted to the considered filtration.

4.2 The Itô Or Stochastic Calculus

In (4.2), the equation is in fact a *stochastic differential equation* or an *Itô differential equation* as the term $dB(t)$ must be considered formally since we know that the sample paths of a Brownian motion are a.s. non-differentiable (see Chapter 1, **Proposition 9.1**).
That is why Itô (1944) created a new type of calculus, called *stochastic calculus* in which the integral with respect to *b* is defined as follows for every stochastic process $f = \left(f(t), t \in [0,T]\right)$ adapted and integrable:

$$\int_0^t f(t,\omega)dB(t,\omega) = \lim_{k\to\infty} \sum_{k=0}^{n-1} f(t_k,\omega)\left[B(t_{k+1},\omega) - B(t_k,\omega)\right], \tag{4.3}$$

where $\left(t_0, t_1, ..., t_n\right), (t_0 = 0, t_n = t, t \in [0,T])$ is a subdivision of $[0,t]$ whose norm tends to 0 for *n* tending to $+\infty$, the limit being the so-called *uniform convergence in probability* (see Protter (1990)).
Conversely, using the differential notation, if the stochastic process $\xi = \left(\xi(t), t \in [0,T]\right)$ is declared to satisfy the following relation, called the *Itô differential* of ξ :

$$d\xi(t) = a(t)dt + b(t)dB(t); \tag{4.4}$$

then:

$$\xi(t) - \xi(0) = \int_0^t a(s)ds + \int_0^t b(s)dB(s). \tag{4.5}$$

For our applications, the main result is the so-called *Itô's lemma* or the *Itô formula*, which is equivalent to the rule of derivatives for composed functions in the classical differential calculus.

Let f be a function of two non-negative real variables x ,t such that

$$f \in C^0_{\mathbb{R}\times\mathbb{R}^+}, f_x, f_{xx}, f_t \in C^0_{\mathbb{R}\times\mathbb{R}^+}. \tag{4.6}$$

Then the composed stochastic process

$$\left(f(\xi(t),t), t \geq 0\right) \tag{4.7}$$

is also Itô differentiable and its stochastic differential is given by:

$$\begin{aligned} &d\left(f(\xi(t),t)\right) = \\ &\left[\frac{\partial f}{\partial x}(\xi(t),t)a(t) + \frac{\partial f}{\partial t}(\xi(t),t) + \frac{1}{2}\frac{\partial^2}{\partial^2 x^2} f(\xi(t),t)b^2(t)\right]dt \\ &+\frac{\partial f}{\partial x}(\xi(t),t)b(t)dB(t). \end{aligned} \tag{4.8}$$

Remark 4.1 Compared with the classical differential calculus, we know that in this case, we should have:

$$\begin{aligned} d\left(f(\xi(t),t)\right) &= \left[\frac{\partial f}{\partial x}(\xi(t),t)a(t) + \frac{\partial f}{\partial t}(\xi(t),t)\right]dt \\ &+\frac{\partial f}{\partial x}(\xi(t),t)b(t)dB(t). \end{aligned} \tag{4.9}$$

So, the difference between relations (4.8) and (4.9) is the *supplementary term*

$$\frac{1}{2}\frac{\partial^2}{\partial^2 x^2} f(\xi(t),t)b^2(t) \tag{4.10}$$

appearing in (4.8) and which is null iff in two cases:

1) f is a linear function of x,
2) b is identically equal to 0.

Examples

1) For ξ given by:

$$\begin{aligned} d\xi(t) &= dB(t), \\ \xi(0) &= 0. \end{aligned} \tag{4.11}$$

Using notation (4.4), we get:

$$a(t)=0,\ b(t)=1. \tag{4.12}$$

With the aid of the Itô formula, the value of $d\xi^2(t)$ is given by

$$d\,\xi^2(t) = \left[2\xi(t).0 + 0 + \frac{1}{2}.2.1\right]dt + 2\,\xi(t).1.dB(t), \tag{4.13}$$

and so

$$dB^2(t) = dt + 2B(t).dB(t). \tag{4.14}$$

As we can see, the first term is the supplementary term with respect to the classical formula and is called the *drift.*

2) Proceeding as for the preceding example, we get for $de^{B(t)}$:

$$de^{B(t)} = \frac{1}{2}e^{B(t)}dt + e^{B(t)}dB(t). \tag{4.15}$$

Here, the drift is given by the first term of the second member of (4.15).

4.3 The Solution Of The Black-Scholes-Samuelson Model

Let us go back to the model (4.2) given by:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dB(t), \\ S(0) &= S_0. \end{aligned} \tag{4.16}$$

Using the Itô formula for *lnS*(*t*)*,* we get:

$$d\ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dB(t) \tag{4.17}$$

and so by integration:

$$\ln S(t) - \ln S_0 = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t). \tag{4.18}$$

Since for every fixed t, $B(t)$ has a normal distribution with parameters $(0,t)$ - t for the variance - (see Chapter 1, **Definition 9.1**), this last result shows that the r.v. $S(t)/S_0$ has a log-normal distribution with parameters $\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$ and so:

$$\begin{aligned} E\left(\log\frac{S(t)}{S_0}\right) &= \left(\mu - \frac{\sigma^2}{2}\right)t, \\ \operatorname{var}\left(\log\frac{S(t)}{S_0}\right) &= \sigma^2 t. \end{aligned} \tag{4.19}$$

Of course, from result (4.18), we obtain the explicit form of the trajectories of the S-process:

$$S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t} e^{\sigma B(t)}. \tag{4.20}$$

This process is called a *geometric brownian motion.*

The fact of having the log-normality confirms the CRR process at the limit as, indeed, a lot of empirical studies show that, for an efficient market, stock prices are well adjusted with such a distribution.
From properties of the log-normal distribution (see Chapter 1, section 5.4), we obtain:

$$E\left(\frac{S(t)}{S_0}\right)=e^{\mu t},$$
$$\operatorname{var}\left(\frac{S(t)}{S_0}\right)=e^{2\mu t}(e^{\sigma^2 t}-1). \tag{4.21}$$

So, we see that the mean value of the asset at time t is given as if the initial amount S_0 was invested at the non-risky instantaneous interest rate μ and that its value is above or below S_0 following the "hazard" variations modelled with the Brownian motion.
We also see that the variance of $S(t)$ increases with time in conformity with the fact that, for long time periods, variations of the asset are very difficult to predict.
The explicit relation (4.20) can also be written in the form:

$$\frac{S(t)}{S_0 e^{\sigma B(t)}}=e^{\left(\mu-\frac{\sigma^2}{2}\right)t}. \tag{4.22}$$

This allows us to distinguish three cases:

(i) $\mu=\dfrac{\sigma^2}{2}$. (4.23)

If so, the evolution of the asset is that of a pure Brownian exponential.

(ii) $\mu>\dfrac{\sigma^2}{2}$. (4.24)

Here, S(t) will vary faster than the pure Brownian exponential and so, we may expect at certain times large gains but also large losses parallel with the time evolution of the pure brownian exponential

(iii) $\mu<\dfrac{\sigma^2}{2}$. (4.25)

Here, the situation is similar but the evolution is opposed to that of the pure Brownian exponential.
From the second result of (4.21), it is also clear that the expectations of large gains - and losses! - are better for large values of σ; that is why σ is called the *volatility* of the considered asset.
It follows that a market with high volatility will attract *risk lover* investors and not *risk averse* investors
From the explicit form, it is not difficult to simulate trajectories of the S-process. The next figure shows a typical form.

Figure 4.1: a typical trajectory

4.4 Pricing The Call With The Black-Scholes-Samuelson Model

4.4.1 The Hedging Portfolio

The problem consists in pricing the value of a European call of maturity T and exercise price K at every time t belonging to $[0,T]$ as a function of t or the maturity at time t, $\tau = T - t$, and of the value of the asset at time t, $S=S(t)$ knowing that the non-risky instantaneous interest rate is r, so that if i is the non-risky annual rate, we have:

$$e^r = 1 + i \,. \tag{4.26}$$

We will use the notation $C(S,t)$ or, more frequently, $C(S,\tau)$.

As in the CRR model, we introduce a portfolio P containing, at every time t of a call and a proportion α, which may be negative, shares of the underlying asset. The stochastic differential of $P(t)$ is given by:

$$dP(t) = dC(S,t) + \alpha dS(t) \tag{4.27}$$

or, from relation (4.16):

$$dP(t) = dC(S,t) + \alpha\mu S(t)dt + \alpha\sigma S(t)dB(t). \tag{4.28}$$

Using the Itô formula, in a correct form as proved by Bartels (1995) of the first initial form given by Black and Scholes (1973), we get:

$$\begin{aligned} dP(t)) &= \left[\frac{\partial C}{\partial S}(S,t)\mu S + \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2 + \alpha\mu S(t) \right] dt \\ &+ \left[\alpha\sigma S(t) + \frac{\partial C}{\partial S}(S,t)\sigma S \right] dB(t). \end{aligned} \tag{4.29}$$

Now, using the principle of AOA, this variation must be identical to that of the same amount invested at the non-risky interest, that is:

$$rP(t)dt = r\left[C(S,t) + \alpha S\right]dt. \tag{4.30}$$

So, we get the following relation:

$$rP(t)dt = dP(t) \tag{4.31}$$

$$\begin{aligned} & r\left[C(S,t) + \alpha S\right]dt = \\ & \left[\frac{\partial C}{\partial S}(S,t)\mu S + \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2 + \alpha\mu S(t)\right]dt \\ & + \left[\alpha\sigma S(t) + \frac{\partial C}{\partial S}(S,t)\sigma S\right]dB(t). \end{aligned} \tag{4.32}$$

By identification, we get:

$$\begin{aligned} & r\left[C(S,t) + \alpha S\right]dt - \\ & \left[\frac{\partial C}{\partial S}(S,t)\mu S + \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2 + \alpha\mu S(t)\right]dt = 0, \\ & \left[\alpha\sigma S(t) + \frac{\partial C}{\partial S}(S,t)\sigma S\right] = 0. \end{aligned} \tag{4.33}$$

From the last equality, we get:

$$\alpha = -\frac{\partial C}{\partial S}(S,t). \tag{4.34}$$

Substituting this value in the first equality of (4.33), we get after simplification:

$$r\left[C(S,t) - \frac{\partial C}{\partial S}(S,t)S\right] - \left[\frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2\right] = 0, \tag{4.35}$$

or finally

$$-rC(S,t) + r\frac{\partial C}{\partial S}(S,t)S + \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\frac{\partial^2 C}{\partial^2 S^2}(S,t)\sigma^2 S^2 = 0, \tag{4.36}$$

a *linear partial derivative equation of order* 2 for the unknown function $C(S,t)$ with as initial condition

$$C(S,t) = \begin{cases} 0, t \in [0,T), \\ \max\{0, S-K\}, t = T \end{cases} \tag{4.37}$$

Using results from the heat equation in physics, for which an explicit solution is given in terms of a so-called Green function, known in this case, Black and Scholes (1973) got the following explicit form for the call value:

$$\begin{aligned} & C(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \\ & d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)\right], \\ & d_2 = d_1 - \sigma\sqrt{T-t}, \\ & S = S(t). \end{aligned} \tag{4.38}$$

Remark 4.2 Using relation (4.31), we get relation (3.15) for t=0 or $\tau = T$. The interpretation is of course already given in section 3.

4.4.2 The Risk Neutral Measure And The Martingale Property

As for the CRR model, it is possible to construct another probability measure *Q* on $\left(\Omega, \Im, (\Im_t)\right)$, called the *risk neutral measure*, such that the value of the call given by formula (4.38) is simply the expectation value of the present value of the "gain" at maturity time *T*.
Using a change of probability measure for going from *P* to *Q*, with the famous Girsanov theorem (see for example Gikhman and Skorokhod,vol.III (1975),p.250), it can be shown that the new measure *Q,* which moreover is unique*,* can be defined by replacing in the stochastic differential equation (4.16) the trend μ by *r.*
Doing so, the explicit form of *S*(*t*) given by relation (4.26) becomes:

$$S(t) = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t} e^{\sigma B'(t)} \tag{4.39}$$

where the process B' still an adapted standard Brownian motion and the value of *C* can be computed as the present value of the expectation of the final "gain" of the call at time *T*:

$$C(S,t) = e^{-r(T-t)} E_Q\left(\sup\left\{S(T) - K, 0\right\}\right). \tag{4.40}$$

The risk neutral measure gives another important property for the process of present values of the asset values on [0,*T*]:

$$\left\{e^{-rt} S(t), t \in [0,T]\right\}. \tag{4.41}$$

Indeed, under *Q*, this process is a martingale, so that (see Chapter 1, section 8) for all *s* and *t* such that $s < t$:

$$E\left(e^{-rt} S(t) \middle| \Im_s\right) \ (s < t) = S(s). \tag{4.42}$$

This means that at every time *s*, the best statistical estimation of *S*(*t*) is given by the observed value at time *s*, a result consistent with the assumption of an efficient market.
From relation (4.42), we get in particular:

$$E\left(e^{-rt} S(t)\right) = S_0. \tag{4.43}$$

So, on average, the present value of the asset at any time *t* equals its value at time 0.
In conclusion, we see that the knowledge of the risk neutral measure avoids the resolution of the partial derivative equation and replaces it by the computation of an expectation, which is in general easier, as it only uses the marginal distribution of *S(*T).

But we must add that, for more complicated derivative products, it may be more interesting, from the numerical point of view, to solve this partial derivative equation with the finite difference method, and particularly in the case of American options.

4.4.3 The Call-Put Parity Relation

From section 1, we know that the value of a put at maturity time T and exercise price K is given by:

$$P\left(S(T),K\right)=\max\left\{0,K-S(T)\right\}. \tag{4.44}$$

As for the call, we have:

$$C\left(S(T),K\right)=\max\left\{0,S(T)-K\right\}, \tag{4.45}$$

and so, we get:

$$C\left(S(T),K\right)-P\left(S(T),K\right)=S(T)-K. \tag{4.46}$$

And so, for the expectations:

$$E\left(C\left(S(T),K\right)\right)-E\left(P\left(S(T),K\right)\right)=E\left(S(T)\right)-K. \tag{4.47}$$

Using the principle of mathematical expectation for pricing the call and the put, we get:

$$e^{rT}C(S_0,0)-e^{rT}P(S_0,0)=E\left(S(T)\right)-K. \tag{4.48}$$

We call this relation the *general call-put parity relation* as it gives the value of the put knowing the value of the call and vice versa

Now, under the assumption of an efficient market, we can use property (4.43) to get

$$e^{rT}C(S_0,0)-e^{rT}P(S_0,0) = S_0e^{rT} - K \tag{4.49}$$

and so the put value is given by:

$$P(S_0,0)=C(S_0,0)-S_0+e^{-rT}K. \tag{4.50}$$

Remark 4.3 We can interpret this relation as follows: assume a portfolio having at time 0 a share of value S_0, a put on the same asset with maturity T and an exercise price K and a sold call with the same maturity and exercise price; the value of the portfolio at time T is always K, whatever the value of $S(T)$ is.

From the call-put parity relation, we easily get the value of a put having the same maturity time T and exercise price K as for the call:

$$P(S,t)=C(S,t)-S+e^{-r(T-t)}K, \tag{4.51}$$

and using the Black and Scholes result, we obtain:

$$
\begin{aligned}
&P(S,t) = Ke^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1),\\
&d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)\right],\\
&d_2 = d_1 - \sigma\sqrt{T-t},\\
&S = S(t).
\end{aligned} \tag{4.52}
$$

5 EXERCISE ON OPTION PRICING

Exercise 5.1 Let us consider a portfolio with Δ shares of unit price 1000 Euro and an amount *B* invested at the non-risky interest rate of 4% per period.

1°) What is the price *C* of a European call having 1050 Euro as exercise price, of maturity 2 periods if per period, the share increases by a quarter of its value with probability 0.75 and decreases by a third of its value with probability 0.25?
What are the intermediate values of the call?

2°) What is the composition of the hedging portfolio at time 0?

3°) If the maturity has for value 2 weeks and the period is the day, give an estimation of the volatility and the trend of the considered asset.

Solution:

1°)
$C_{uu} = 512.5, C_{ud} = C_{dd} = 0,$
$C_u = 315.38, C_d = 0,$
$C = 194.08.$

2°)
$C = \Delta S + B$ where:

$\Delta = \dfrac{C_u - C_d}{S(u-d)} = 54.07\%($ part of the asset),

$B = \dfrac{uC_d - dC_u}{(u-d)} = -346.57F$(loan at the non-risky rate from the bank).

3°)
We know that:

$$1000\times\frac{5}{4}=1000\times e^{\mu\frac{t}{n}+\sigma\sqrt{\frac{t}{n}}},$$
$$1000\times\frac{2}{3}=1000\times e^{\mu\frac{t}{n}-\sigma\sqrt{\frac{t}{n}}},$$
$$or:$$
$$t=14\ days,$$
$$n=1\ day,$$
$$so:$$
$$\frac{5}{4}=e^{\mu 14+\sigma\sqrt{14}}\Rightarrow 14\mu+\sqrt{14}\sigma=\ln\frac{5}{4},$$
$$\frac{2}{3}=e^{\mu 14-\sigma\sqrt{14}}\Rightarrow 14\mu-\sqrt{14}\sigma=\ln\frac{5}{4}.$$
$$Finally, we\ get:$$
$$\mu=\frac{1}{28}0.2231436=0.0079694,$$
$$\mu_{year}=360\times 0.0079694=2.868994,$$
$$\sigma=\frac{1}{2\sqrt{14}}0.2231436=0.0298188,$$
$$\sigma_{year}=\sqrt{360}.0.0298188=0.565772.$$

6 THE GREEK PARAMETERS

6.1 Introduction

The technical management of the trader of options, particularly by the brokers, uses the so-called *Greek parameters* to measure the impacts of small variations of parameters involved in formulas (4.38) and (4.52) for the pricing of options: S,σ,τ,r,K.

(i) The delta coefficient
This is an indicator concerning the influence of small variations ΔS of the asset price defined as follows:

$$C(S+\Delta S,t)\approx C(S,t)+\Delta(\Delta S),$$
$$\Delta=\frac{\partial C}{\partial S}(S,t). \tag{6.1}$$

This parameter is often used to cancel the variations of the asset value in the hedging portfolio.

(ii) The gamma coefficient
It is defined as:

$$\gamma = \frac{\partial^2 C}{\partial S^2}(S,t) \tag{6.2}$$

and so it may be seen as the *delta of the delta.*
It gives a measure of the acceleration of the variation of the call and a refinement of the measure of the variation of the call using the Taylor formula of order 2:

$$C(S+\Delta S,t) \approx C(S,t) + \Delta\Delta t + \frac{1}{2}\gamma\Delta t^2. \tag{6.3}$$

(iii) The theta coefficient
It gives the dependence of C with respect to the maturity $\tau(=T-t)$, and so also from the time t:

$$\theta = -\frac{\partial C}{\partial t}\left(=\frac{\partial C}{\partial \tau}\right). \tag{6.4}$$

It follows the first order approximation:

$$C(S,t+\Delta t) \approx C(S,t) - \theta\Delta t. \tag{6.5}$$

For the maturity variations $\tau = T-t$, we get:

$$C(S,\tau+\Delta\tau) \approx C(S,\tau) + \theta\Delta\tau. \tag{6.6}$$

(iv) The elasticity coefficient
Recall the economic definition of this coefficient which gives here:

$$e(S,t) = \frac{\partial C}{\partial S}(S,t) \times \frac{S}{C(S,t)} \tag{6.7}$$

and so:

$$\frac{\Delta C}{C}\left(=\frac{C(S+\Delta S,t)-C(S,t)}{C(S,t)}\right) \approx e(S,t)\frac{\Delta S}{S}. \tag{6.8}$$

(v) The vega coefficient
It is the indicator concerning the measure of small variations of the volatility σ and so:

$$\upsilon = \frac{\partial C}{\partial \sigma}(S,t). \tag{6.9}$$

Thus, we have approximately for small variations $\Delta\sigma$,

$$C(S+\Delta S,t) \approx C(S,t) + \upsilon\Delta o. \tag{6.10}$$

(vi) The rhô coefficient
It concerns the non-risky instantaneous rate r and so:

$$\rho = \frac{\partial C}{\partial r}(S,t). \tag{6.11}$$

6.2 Values Of The Greek Parameters

The following table gives the values of the Greek parameters first for the call and then for the put.

I.For the calls:

1)delta$(=\frac{\partial \mathrm{C}}{\partial \mathrm{S}})=\Phi(d_1) > 0$

2)gamma$(=\frac{\partial \Delta}{\partial \mathrm{S}}) = \frac{\Phi'(d_1)}{S\sigma\sqrt{\tau}} > 0$

3)véga$(=\frac{\partial \mathrm{C}}{\partial \sigma})=S\sqrt{\tau}\Phi'(d_1) > 0$

4)rhô$(=\frac{\partial \mathrm{C}}{\partial \mathrm{r}})=K\tau \mathrm{e}^{-\mathrm{r}\tau}\Phi(d_2) > 0$

5)théta$(=\frac{\partial \mathrm{C}}{\partial \tau})=\mathrm{rKe}^{-\mathrm{r}\tau}\Phi(d_2)+\frac{\sigma \mathrm{S}}{2\sqrt{\tau}}\Phi'(d_1) > 0$

6)$\frac{\partial C}{\partial K} = -e^{-r\tau}\Phi(d_2) < 0$

II.For the puts:

1)delta$(=\frac{\partial \mathrm{P}}{\partial \mathrm{S}})=(\Phi(d_1)-1) = -\Phi(-d_1)(=\Delta_C - 1) < 0$

2)gamma$(=\frac{\partial \Delta}{\partial \mathrm{S}}) = \frac{\Phi'(d_1)}{S\sigma\sqrt{\tau}}(= gamma_C) > 0$

3)véga$(=\frac{\partial \mathrm{P}}{\partial \sigma})=S\sqrt{\tau}\Phi'(d_1)(= véga_C) > 0$

4)rhô$(=\frac{\partial \mathrm{P}}{\partial \mathrm{r}})=-\tau \mathrm{K}e^{-r\tau}\Phi(-d_2) = \tau \mathrm{K}e^{-r\tau}\left[\Phi(d_2)-1\right](= rh\hat{o}_C - \tau \mathrm{K}e^{-r\tau}) < 0$

5)théta$(=\frac{\partial \mathrm{P}}{\partial \tau})=\frac{\sigma S}{2\sqrt{\tau}}\Phi'(d_1) - r\mathrm{Ke}^{-\mathrm{r}\tau}\left[1-\Phi(d_2)\right](= \theta_C - rKe^{-r\tau})$

6)$\frac{\partial P}{\partial K} = e^{-r\tau}(-\Phi(d_2)+1) = e^{-r\tau}\Phi(-d_2)(= \frac{\partial P}{\partial K}_C + e^{-r\tau}) > 0$

These values give interesting results concerning the influence of the considered parameters of the call and put values.
For example, we deduce that the call and put values are increasing functions of the volatility, and the call increases as S increases but the put decreases as S increases.

6.3 Exercises

Exercise 6.1
Let us consider an asset of value 1700 Euro and having as weekly variance 0.000433.
(i) What is the value of a call of exercise price 1750 Euro with maturity 30 weeks under a non-risky rate of 6%?
(ii) Under the anticipation of a rise of 100 Euro of the underlying asset and of a rise of 0.000018 of the weekly variance, what will be the consequences of the call and put values?

Solutions:
(i) The values of the parameters necessary to compute the call value using the Black and Scholes formula are:

$$\sigma^2_{week.} = 0.00043 \Rightarrow \sigma^2_{year.} = 52 \times 0.00043 = 0.2236, \sigma_{year.} = 0.47286,$$

$$\tau = 30\ weeks. = 0.576923\ year, K = 1750, S = 1700,$$

$$i = 6\% \Rightarrow r = \ln(1+i) = 0.05827.$$

It follows that:

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{S}{K} + \tau(r + \frac{\sigma^2}{2})\right] \Rightarrow d_1 = 0.09760272,$$

$$\Phi(d_1) = 0.5388762,$$

$$\Rightarrow d_2 = d_1 - \sigma\sqrt{\tau} = -0.01637096, \Phi(d_2) = 0.4934692,$$

$$C(S,\tau) = S\Phi(d_1) - Ke^{-r\tau} = 81.07 Euro.$$

Using call-put parity relation; we get for the put value

$$P = Ke^{-r\tau} + C - S \Rightarrow P = 73.07 Euro.$$

(ii) *Rise of the underlying asset:*
We know that:

$$C(S+\Delta S,\tau)=C(S,\tau)+\frac{\partial C}{\partial S}(S,\tau)\Delta S,$$

$$\frac{\partial C}{\partial S}(S,\tau)=\Phi(d_1),$$

so:

$$C(1700+100,\tau)=81.07+100\times 0.5388762=135.95Euro.$$

For the put, we obtain:

$$P(S+\Delta S,\tau)=Ke^{-r\tau}+C(S+\Delta S)-(S+\Delta S)=27.1Euro.$$

(iii) *Rise of the volatility:*

The value of the new weekly variance is now given by:

$$0.000433+0.00018=0.000613$$

and so the new yearly variance and volatility are given by

$$\sqrt{0.031876}=0.1785385,$$

and consequently, the variation of the yearly volatility is given by:

$$\Delta\sigma=0.1785385-0.1500533=0.284852.$$

As the increase in volatility comes after that of the asset value, we have

$$C(S+\Delta S,\sigma+\Delta\sigma,\tau)=C(S+\Delta S,\sigma,\tau)+\frac{\partial C}{\partial\sigma}\Delta\sigma,$$

with:

$$\frac{\partial C}{\partial\sigma}=\sqrt{\tau}\Phi'(d_1).$$

But:

$$\Phi'(d_1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}=0.39704658,$$

and so:

$$\frac{\partial C}{\partial\sigma}=542.84.$$

Finally, we get:

$$C(S+\Delta S,\sigma+\Delta\sigma,\tau)=C(S+\Delta S,\sigma,\tau)+\frac{\partial C}{\partial\sigma}\Delta\sigma=150.41F.$$

For the variation for the put, we use the call-put parity relation and so:

$$P(S+\Delta S,\sigma+\Delta\sigma,\tau)=C(S+\Delta S,\sigma+\Delta\sigma,\tau)+Ke^{-r\tau}-(S+\Delta S)=42.56F.$$

Exercise 6.2

For the following data, compute the values of the call and the put and the Greek parameters

$S=100, K=98, \tau=30$ days, $\sigma_{week}=0{,}01664, i=8\%.$

Solution

Yearly vol.	0.12	
Maturity.	0.08219	
r=ln(1+i)	0.076962	
Results	**call**	**put**
Price	3.04721	0.42926
Delta	0.7847	-0.2153
Vega	8.3826	idem
Theta	11.924	4.334
Gamma	0.08499	idem
Rhô	6.199	-1.805

Table 6.1: example option computation

7 THE IMPACT OF DIVIDEND DISTRIBUTION

If between t and T, the asset distributes N dividends of amounts $D_1,...,D_N$ at times:

$$(0 < t <)t_1 < t_2 < \cdots < t_N (< T)\,, \tag{7.1}$$

the impact of the value of a European call is the following : as the buyer of the call cannot receive these dividends, it suffices to compute the present value at time t of these dividends and to subtract the sum from the asset value at time t so that the call value is now:

$$C(S,\tau;D_1,...,D_N) = C(S - \sum_{j=1}^{N} D_j e^{-r\tau_j}, \tau), \tag{7.2}$$
$$\tau_j = t_j - t, j = 1,...,N.$$

Of course, the most usual case is N=1.
If we assume that the distribution of dividends is given with a continuous payout at rate D per unit of time,

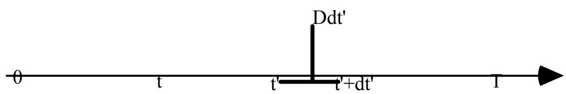

Figure 7.1: continuous "payout"

the capitalised value is $e^{D\tau}$ and so the value of the call is given by:

$$C(S,\tau;D) = C(Se^{-Dt},\tau). \tag{7.3}$$

8 ESTIMATION OF THE VOLATILITY

8.1 Historic Method

This method is based on the data of the underlying asset evolution in the past, for example the n daily values

$$\left(S_0 S_1,...,S_n\right). \tag{8.1}$$

Let us consider the following sample of the consecutive ratios:

$$\left(R_1,...,R_n\right) = \left(\frac{S_1}{S_0},...,\frac{S_n}{S_{n-1}}\right). \tag{8.2}$$

From the log-normal distribution property, we have:

$$\frac{\ln R_t - (\mu - \frac{\sigma^2}{2})}{\sigma} \succ N(0,1), \tag{8.3}$$

$$\text{with } R_t = \frac{S_t}{S_{t-1}}, t = 1,...,n.$$

It follows that the random sample $\left(\ln R_1,...,\ln R_n\right)$ can be seen as extracted from a normal population $\left(\mu',\sigma^2\right)$ with:

$$\mu' = \mu - \frac{\sigma^2}{2}. \tag{8.4}$$

The classical results of mathematical statistics give as best estimators:

$$\hat{\mu}' = \frac{1}{n}\sum_{k=1}^{n} \ln\frac{R_k}{R_{k-1}},$$
$$\hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{n}\left(\ln\frac{R_k}{R_{k-1}} - \hat{\mu}'\right)^2. \tag{8.5}$$

To get an unbiased estimator of the variance, we have to use:

$$\hat{\hat{\sigma}}^2 = \frac{n}{n-1}\hat{\sigma}^2 \tag{8.6}$$

or:

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(\ln\frac{R_k}{R_{k-1}}\right)^2 - \frac{n}{n-1}(\hat{\mu}')^2. \tag{8.7}$$

Example 8.1 On the basis of a sample of 27 weekly values of an asset starting from the initial value 26.375 Euro, the following *weekly* estimations are found:
$\hat{\mu} = 0.016732,$

$\hat{\sigma}^2 = 0.005216.$
Consequently, as the parameters of the Black and Scholes model must be evaluated on a yearly basis, we get
$\hat{\mu}_{an.} = 52 \times 0.016732 = 0.870064 \cong 0.87,$

$\hat{\sigma}^2_{an.} = 52 \times 0.005216 = 0.271232,$

$\hat{\sigma}_{an.} = \sqrt{0.271232} = 0.520799 \cong 0.52.$

8.2 Implicit Volatility Method

This method assumes that the Black and Scholes formula calibrates the market values of the observed calls well.
Theoretically, an inversion of the Black and Scholes formula gives the value of the volatility σ.
On the basis of several observations of the calls for the same underlying asset, we can use the least square statistical method to refine the estimation.

Example 8.2
Using the data of **Excercise 6.2**, we assume that we have an observed value of the call 3.04715, but without knowing the volatility.
The next table gives the results using a step by step approximation method.

weekly vol.	**annual vol.**	**call value**
0.02	0.144	3.26
0.015	0.1081	2.95
0.017	0.1225	3.069
0.016	0.1153	3.008
0.0165	0.1189	3.038
0.01664	0.1199	3.04713

Table 8.1: volatility computation

So, we do find the correct volatility value 0.12.

Remark 8.1 The main difficulty is to select the historical data.
The set must not be too long or too short in order to avoid perturbed periods introducing strong biases in the results.

Moreover, we always work with the assumption of a constant volatility that we will suppress in section 10.

9 BLACK AND SCHOLES ON THE MARKET

9.1 Empirical Studies

Since the opening of the CBOT in Chicago in 1972, numerous studies have been carried out for testing the results of the Black and Scholes formula.
In the case of efficient markets, the conclusions are the following:

(i) the non-risky interest rate has little influence on the option values,
(ii) the Black and Scholes formula *underestimates* the market values for calls with short maturity times, for calls "deep out of the money" ($S/K<0.75$) and for calls with weak volatility,
(iii) the Black and Scholes formula *overestimates* the market values for calls "deep in the money" ($S/K<1.25$) and for calls with high volatility. The put values are often underestimated particularly in the "out of the money" ($S>>K$) case.
(iv) the puts are often underestimated particularly when they are out of the money($S<<K$).

9.2 Smile Effect

If we compute the volatility values with the implicit method in different times, in general, the results show that the volatility is *not constant*, invalidating thus one of the basic assumptions of the considered Black and Scholes model.
The graph of the volatility as a function of the exercise price often gives a graph with a convex curve, a result commonly called the "*smile effect*".
But sometimes, concave functions are also observed.
Although, theoretically, volatilities for the pricing of calls and puts are identical, in practice, some differences are observed; they are assigned to differences of "bid-offer spread" and to the methodology of the implicit method used at different times.
The fact that it is important to consider option pricing models with non-constant volatility is one of the motivations of the next model.

10 THE JANSSEN-MANCA MODEL

In this section, we present a new extension of the fundamental Black and Scholes (1973) formula in stochastic finance with the introduction of a random economic

and financial environment using Markov processes and which we owe to Janssen and Manca (1999).
In preceding papers (Janssen et al (1995), Janssen et al (1997), Janssen et al (1998), Janssen and Manca (2000)), these authors already show how it is useful to introduce Markov and semi-Markov theory in finance, with the assumption that the evolution of the asset follows a semi-Markov process, homogeneous or non-homogeneous, and how to price options in such new models. The main idea of this approach is to insert a strong dependence of the asset evolution as a function of the preceding value.
The construction of this new model starts from the classical CRR model with one period to obtain a new continuous time model satisfying the assumption of absence of arbitrage.
One of the main potential applications of our model concerns the possibility to get a new way of acting with the Black and Scholes formula with information related to the economic and financial environment, particularly concerning the volatility of the underlying asset.
This new model also gives the possibility to take into account *anticipations* of investors in such a way as to incorporate them in their own option pricing.
By the same philosophy, the model can be used to construct scenarios and particularly in the case of stress in a VaR approach.

10.1 The Markov Extension Of The One-Period CRR Model

10.1.1 The Model

Starting on a complete probability space $(\Omega, \Im, P)$, let us consider a one-period model for the evolution of one asset having the known value $S(0) = S_0$ at time 0 and random value $S(1)$ at time 1.
The economic and financial environment is defined with random variables J_0, J_1 representing the environment states respectively at time 0 and time 1. These random variables take their values in the state space $E = \{1,...,m\}$ and are defined on the probability space by:

$$\begin{aligned} &P(J_0 = i) = a_i, i = 1,...,m; \\ &P(J_1 | J_0 = i) = p_{ij}, i, j = 1,...,m, \end{aligned} \tag{10.1}$$

where:

$$\begin{aligned} &a_i \geq 0, i = 1,...,m; \\ &\sum_{i=1}^{m} a_i = 1, \\ &p_{ij} \geq 0, i, j = 1,...,m, \\ &\sum_{j=1}^{m} p_{ij} = 1,\ i = 1,...,m. \end{aligned} \tag{10.2}$$

Furthermore, let us introduce the following function of J_0, J_1: $u_{J_0J_1}, d_{J_0J_1}$, $q_{J_0J_1}$ such that, a.s.:

$$\begin{aligned} &0 < d_{J_0J_1} < r_{J_0J_1} < u_{J_0J_1}, \\ &d_{J_0J_1} < 1,\ 1 < r_{J_0J_1}, \end{aligned} \tag{10.3}$$

$$0 < q_{J_0J_1} < 1. \tag{10.4}$$

The *one-period model*, related to the process $\{S(0), S(1)\}$, is the following: given J_0, J_1 and that $S(0) = S_0$, the asset has the following evolution: it goes up from S_0 to $u_{J_0J_1}S_0$ with the conditional probability $q_{J_0J_1}$ or goes down from S_0 to $d_{J_0J_1}S_0$ with the conditional probability $1 - q_{J_0J_1}$; moreover, the non-risky interest rate of this period has the value $\nu_{J_0J_1}$ defined by:

$$\nu_{J_0J_1} = r_{J_0J_1} - 1. \tag{10.5}$$

Given J_0, J_1, we have that:

$$\begin{aligned} &P\left(S(1) = u_{J_0J_1}S_0 \middle| J_0, J_1, S_0\right) = q_{J_0J_1}, \\ &P\left(S(1) = d_{J_0J_1}S_0 \middle| J_0, J_1, S_0\right) = 1 - q_{J_0J_1}, \\ &E\left(S(1) \| J_0, J_1, S_0\right) = q_{J_0J_1}u_{J_0J_1}S_0 + (1 - q_{J_0J_1})d_{J_0J_1}S_0, \\ &E\left(S(1) \| J_0, S_0\right) = \sum_{j=1}^{m} p_{J_0j}(q_{J_0j}u_{J_0j}S_0 + (1 - q_{J_0j})d_{J_0j}S_0), \\ &E\left(S(1) \| S_0\right) = \sum_{i=1}^{m} P\left(J_0 = i\right) \sum_{j=1}^{m} \left[p_{ij}(q_{ij}u_{ij} + (1 - q_{ij})d_{ij})\right] S_0. \end{aligned} \tag{10.6}$$

One of the basic concepts of stochastic finance is the *absence of arbitrage possibility*. In fact, it is equivalent to say that the process $\{r^{-1}S(0), S(1)\}$ is a martingale where $r = 1 + \rho$ and ρ is an adequate non-risky interest rate for computing the present value of $S(1)$ at time 0.

Here, we must take into account the possible information of the investor concerning the environment; at time 0, in addition to the knowledge of S_0, different information sets may be available. Three cases are possible:

1) *knowledge of* $\left(J_0, J_1\right)$

In this case the martingale condition:

$$E\left(S(1)\|J_0, J_1, S_0\right) = r_{J_0J_1}S_0 \tag{10.7}$$

becomes:

$$r_{J_0J_1}S_0 = q_{J_0J_1}u_{J_0J_1}S_0 + (1-q_{J_0J_1})d_{J_0J_1}S_0 \tag{10.8}$$

or

$$r_{J_0J_1} = q_{J_0J_1}u_{J_0J_1} + (1-q_{J_0J_1})d_{J_0J_1}. \tag{10.9}$$

This last condition is exactly the same as the CRR model; this means that the new conditional probability for which the martingale condition is satisfied is given by:

$$\tilde{q}_{J_0J_1} = \frac{r_{J_0J_1} - d_{J_0J_1}}{u_{J_0J_1} - d_{J_0J_1}}. \tag{10.10}$$

This value defines the so-called *risk neutral conditional probability measure.*

As an example of application in *option pricing*, let us consider that we want to study a European call option of maturity T=1 and exercise price K bought at time 0.

It follows that at time 1 or at the end of the maturity, the value of the option will be given by the random variable:

$$C(S(1),0) = \max\left\{0, S(1)-K\right\}. \tag{10.11}$$

We compute the price of the option at time 0 with a maturity period of value 1, as the conditional expectation under the risk neutral conditional probability measure, denoted $C_{J_0,J_1}(S_0,1)$, of the present value of the gain at time 1:

$$\begin{aligned} C_{J_0,J_1}(S_0,1) &= E\left(r_{J_0J_1}^{-1}\max\left\{0,S(1)-K\right\}\middle|J_0,J_1\right) \\ &= r_{J_0J_1}^{-1}\left[\tilde{q}_{J_0J_1}\max\left\{0,u_{J_0J_1}S_0-K\right\} + (1-\tilde{q}_{J_0J_1})\max\left\{0,d_{J_0J_1}S_0-K\right\}\right]. \end{aligned} \tag{10.12}$$

2) *knowledge of* J_0

Let us begin to see what the martingale condition becomes.

We have that:

$$E\left(S(1)\|J_0,S_0\right) = E\left(E\left(S(1)\|J_0,J_1,S_0\right)\middle|J_0,S_0\right). \tag{10.13}$$

As the assumption of AOA is now satisfied for the conditioning with J_0, J_1, we can write that

$$E\left(S(1)\|J_0,S_0\right) = E\left(r_{J_0J_1}S_0\middle|J_0,S_0\right), \tag{10.14}$$

and so:

$$E\left(S(1)\|J_{0,},S_0\right) = S_0E\left(r_{J_0J_1}\middle|J_0,S_0\right), \tag{10.15}$$

and finally:

$$E\left(S(1)\middle\|J_{0,},S_0\right)=\varsigma_{J_0}S_0 \tag{10.16}$$

where:

$$\varsigma_{J_0}=\sum_{j=1}^{m}p_{J_0j}r_{J_0j}\,. \tag{10.17}$$

These last two formulas show that, given, at time 0, the initial environment state, the AOA is still valid with as risk neutral interest

$$\rho_{J_0}=1-\varsigma_{J_0}, \tag{10.18}$$

or

$$\rho_{J_0}=\sum_{j=1}^{m}p_{J_0j}\nu_{J_0j}, \tag{10.19}$$

with r_{J_0j} given by relation (10.5) which is perfectly coherent as relation (10.19) represents the conditional mean of the non-risky interest rate given J_0.

3) *no environment knowledge*

In this last case, the investor just observes the initial value of the stock S_0 as in the CRR or the Black and Scholes models. As above we can compute the expectation of $S(1)$ as follows:

$$E\left(S(1)\middle|S_0\right)=E\left(E\left(S(1)\middle|J_0\right)\middle|S_0\right), \tag{10.20}$$

and from relation (10.16):

$$E\left(S(1)\middle|S_0\right)=S_0E\left(\varsigma_{J_0}\middle|S_0\right). \tag{10.21}$$

As, from relation (10.17), we get that:

$$E\left(\varsigma_{J_0}\middle|S_0\right).=\sum_{i=1}^{m}a_i\sum_{j=1}^{m}p_{ij}r_{ij}\ , \tag{10.22}$$

it follows that the AOA is still true in this case with a non-risky interest rate ρ defined by:

$$\rho=1-\sum_{i=1}^{m}a_i\sum_{j=1}^{m}p_{ij}r_{ij}. \tag{10.23}$$

From this last relation and relation (10.19), we get

$$\begin{aligned}\rho&=\sum_{i=1}^{m}a_{\mathrm{i}}-\sum_{i=1}^{m}a_i\sum_{j=1}^{m}p_{ij}(1-\nu_{ij})\\&=\sum_{i=1}^{m}a_i\sum_{j=1}^{m}p_{ij}\nu_{ij}\\&=\sum_{i=1}^{m}a_i\nu_i.\end{aligned} \tag{10.24}$$

Once more, these last two relations show the perfect coherence concerning the non-risky interest rates to be used with regard to the three environment information sets that we can have.

10.1.2 Computational Option Pricing Formula For The One-Period Model

In the preceding section, relation (10.12) gives the value of a call option at time 0 given the initial and final environment states J_0, J_1. We now compute the price of the option, firstly with only the knowledge at time 0 of the initial environment state J_0, then with only the knowledge of the final state J_1 and finally with no knowledge of the initial and final states.

1) *with the knowledge of* J_0

This value, denoted by $C_{J_0}(S_0,1)$, is nothing else than the conditional expectation of $C_{J_0J_1}(S_0,1)$ given J_0:

$$C_{J_0}(S_0,1) = E\left(C_{J_0J_1}(S_0,1) \middle| J_0, S_0\right), \tag{10.25}$$

or

$$C_{J_0}(S_0,1) = \sum_{j=1}^{m} p_{J_0 j} C_{J_0 j}(S_0,1). \tag{10.26}$$

2) *with the knowledge of* J_1

Let $C^j(S_0,1)$ represent the value of the call in this case when $J_1 = j$; we have:

$$C^j(S_0,1) = \sum_{i=1}^{m} P\left(J_0 = i \middle| J_1 = j\right) C_{ij}(S_0,1). \tag{10.27}$$

From the Bayes formula, we get:

$$\begin{aligned} P\left(J_0 = i \middle| J_1 = j\right) &= \frac{P\left(J_0 = i, J_1 = j\right)}{P\left(J_1 = j\right)} \\ &= \frac{a_i p_{ij}}{\sum_{k=1}^{m} a_k p_{kj}} \end{aligned} \tag{10.28}$$

and so, from relation (10.27):

$$C^j(S_0,1) = \sum_{i=1}^{m} \frac{a_i p_{ij}}{\sum_{k=1}^{m} a_k p_{kj}} C_{ij}(S_0,1). \tag{10.29}$$

Let us note that this case is useful if the investor wants to anticipate the final value of the environment state at time 0.

3) *with no knowledge of* J_0 *and* J_1

In this case, with the help of relation (10.26), we can write that the call value represented by $C(S_0,1)$, is given by:

$$C(S_0,1)=\sum_{i=1}^{m} a_i C_i(S_0,1), \tag{10.30}$$

or with the help of relation (10.29) with:

$$C(S_0,1)=\sum_{j=1}^{m}\sum_{k=1}^{m} a_k p_{kj} C^j(S_0,0). \tag{10.31}$$

10.1.3 Examples

The application of our one-period model is already useful with only two or three states. Indeed, it is quite natural to consider one state, for example state 0 to model the *normal* economic and financial environment; then we can add a supplementary state -1 to represent an *abnormal* situation like a crash or a doped situation.

With three states, we can separate the crash possibility represented by state -1 from the doped situation represented by state 1, state 0 always being the normal case.

Example 10.1: *A two-states model*

As said just above, let the state set be:

$$I=\{0,1\} \tag{10.32}$$

with state 0 as the *normal* economic and financial situation environment and state 1 as the *exceptional* in the sense of, for example, a crash or doped situation.

Numerical data are the following:

$$\begin{gathered}\mathbf{a}=(0.95,0.05),\\ \mathbf{P}=\begin{bmatrix}0.98 & 0.02\\ 0.60 & 0.4\end{bmatrix}, \underline{\mathbf{r}}=\begin{bmatrix}1.03 & 1.05\\ 1.05 & 1.03\end{bmatrix},\\ \underline{\mathbf{U}}=\begin{bmatrix}1.3 & 1.1\\ 1.06 & 1.2\end{bmatrix}, \underline{\mathbf{D}}=\begin{bmatrix}0.7 & 0.5\\ 0.4 & 0.6\end{bmatrix}.\end{gathered} \tag{10.33}$$

Example 10.2: *A three-states model*

Here, let the state set be:

$$I=\{-1,0,1\}. \tag{10.34}$$

State 0 represents the *normal* economic and financial situation environment, state -1 the *exceptionally bad* situation in the sense of for example a crash situation and state 1 as *exceptionally good* as a doped effect of the Stock Exchange for example.

Numerical data are the following:

$$\mathbf{a} = (0.05, 0.90, 0.05),$$

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.02 & 0.96 & 0.02 \\ 0.6 & 0.35 & 0.05 \end{bmatrix}, \underline{\mathbf{r}} = \begin{bmatrix} 1.05 & 1.03 & 1.02 \\ 1.05 & 1.03 & 1.02 \\ 1.06 & 1.04 & 1.03 \end{bmatrix}, \tag{10.35}$$

$$\underline{\mathbf{U}} = \begin{bmatrix} 1.07 & 1.10 & 1.20 \\ 1.07 & 1.10 & 1.20 \\ 1.07 & 1.09 & 1.15 \end{bmatrix}, \underline{\mathbf{D}} = \begin{bmatrix} 0.5 & 0.7 & 0.8 \\ 0.6 & 0.7 & 0.8 \\ 0.65 & 0.7 & 0.8 \end{bmatrix}.$$

For both examples, we will consider a European call option with $S_0 = 100 \; and \; K = 95$.

Results are given in **Table 10.1**.

S	100										
K	95										
Example 10.1											
transition	a1	a2	a3	p(ij)	r(ij)	u(ij)	d(ij)	q(ij)	**Cij(100,1)**	**Ci(100,1)**	**C(100,1)**
0 to 0	0.95	0.05		0.98	1.03	1.3	0.7	0.55	18.68932	18.57744	
0 to 1				0.02	1.05	1.1	0.5	0.9167	13.09524		
1 to 0				0.6	1.05	1.06	0.4	0.9848	10.31746	13.1484	
1 to 1				0.4	1.03	1.2	0.6	0.7167	17.39482		
											18.27158
Example 10.2											
	0.05	0.9	0.05								
bad to bad				0.6	1.05	1.07	0.5	0.9649	11.02757	11.56895	
bad to normal				0.3	1.03	1.1	0.7	0.825	12.01456		
bad to good				0.1	1.02	1.2	0.8	0.55	13.48039		
normal to bad				0.02	1.05	1.07	0.6	0.9574	10.94225	12.02243	
normal to normal				0.96	1.03	1.07	0.7	0.8919	12.01456		
normal to good				0.02	1.02	1.07	0.8	0.8148	13.48039		
good to bad				0.6	1.02	1.2	0.65	0.6727	11.05121	7.67948	
good to normal				0.35	1.02	1.2	0.7	0.64	11.7357		
good to good				0.05	1.03	1.15	0.8	0.6571	12.76006		
											11.82274

Table 10.1: European call option examples

10.2 The Multi-Period Discrete Markov Chain Model

Let us now consider a multi-period model over the time interval $\left[0,n\right]$, n being an integer larger than 1 representing the maturity time of the option, always under the assumption of absence of arbitrage as in section 1.
To obtain useful results, we will still follow the fundamental presentation of the CRR model (Cox, Rubinstein (1985)) but adapted for our Markov environment in such a way that tractable results may be found.

1) *result with knowledge of* $J_0,...,J_n$
Let us begin with a discrete time model with n periods and suppose that given $J_0,...,J_n,S(0)=S_0$ with $J_0=i,J_n=j,$ the up and down parameters, the non-risky interest rate and the probabilities of an up movement for each period are the same for all periods and given respectively by $u_{ij},d_{ij},r_{ij},q_{ij}$.
Then, the asset value $S(n)$ at time n is given by:

$$S(n)=V_{j_0j_1}\cdot\cdots\cdot V_{j_{n-1}j_n}S_0 \qquad (10.36)$$

where the conditional distributions of the random variables V are defined as:

$$V_{J_{n-1}J_n}=\begin{cases} u_{ij} \text{ with probability } q_{ij}, \\ d_{ij} \text{ with probability } 1\text{-}q_{ij}, \end{cases} i,j\in I. \qquad (10.37)$$

Moreover, we suppose that, for each n, the random variables $V_{J_0J_1},...,V_{J_{n-1}J_n}$ are conditionally independent given $J_0,...,J_n$.
Now if the random variable M_n represents the total number of up movements on $\left[0,n\right]$, the asset value at time n is given by:

$$S(n)=(u_{ij})^{M_n}(d_{ij})^{n-M_n}S_0 \qquad (10.38)$$

and consequently:

$$\ln\frac{S(n)}{S_0}=M_n\ln u_{ij}+(n-M_n)\ln d_{ij}. \qquad (10.39)$$

Given $J_0=j_0,...,J_n=j_n,S(0)=S_0$, the conditional distribution of M_n is a binomial distribution with parameters (n,q_{ij}). It follows that:

$$E\left(\ln\frac{S(n)}{S_0}|J_0=j_0,...,J_n=j_n,S(0)=S_0\right)=n(q_{ij}\ln u_{ij}+(1-q_{ij})\ln d_{ij}) .(10.40)$$

Concerning the conditional variance, we get:

$$\text{var}\left(\ln\frac{S(n)}{S_0}|J_0=j_0,...,J_n=j_n,S(0)=S_0\right)=n\left[q_{ij}(1-q_{ij})\left(\ln\frac{u_{ij}}{d_{ij}}\right)^2\right].(10.41)$$

Choosing now for the up probability on the n periods, the risk neutral probability given by relation (10.10):

$$\tilde{q}_{ij_1} = \frac{r_{ij_1} - d_{ij_1}}{u_{ij_1} - d_{ij_1}}, \tag{10.42}$$

it is now clear that, under our assumptions, for each n, given $J_0,...,J_n,S(0)=S_0$ with $J_0 = i,\ J_n = j,$ we have a CRR model, so that their results recalled in the beginning of this chapter concerning the European call are valid. Consequently, we get the value of the European call with exercise price and maturity n as the present value of the expectation of the "gain" at time n under the risk neutral measure, that is:

$$\begin{aligned} C_{ij}(S_0,n) &= C(S_0,n \mid J_0 = i, J_1,...,J_n = j) \\ &= \frac{1}{v_{ij}^n}\sum_{k=0}^{n}\binom{n}{k}\tilde{q}_{ij}^k(1-\tilde{q}_{ij})^{n-k}\max\left\{u_{ij}^k d_{ij}^{n-k}S_0 - K\right\}. \end{aligned} \tag{10.43}$$

After some computation, we can obtain the following expression (see Cox and Rubinstein (1985)):

$$C(S_0,n \mid J_0 = i, J_1,...,J_n = j) = \begin{cases} S_0 B(a_{ij};n,\tilde{q}'_{ij}) - \dfrac{K}{v_{ij}^n}B(a_{ij};n,\tilde{q}_{ij}), & \text{if } a_{ij} < n, \\ 0 & \text{if } a_{ij} > n, \end{cases} \tag{10.44}$$

where $B(x;m,\alpha)$ is the value of the complementary binomial distribution function complementary with parameters m,α at point x and

$$\begin{aligned} a_{ij} &= \left\lceil \frac{\ln(K / d_{ij}^n S_0)}{\ln(u_{ij}/d_{ij})} + 1 \right\rceil, \\ \tilde{q}'_{ij} &= \frac{u_{ij}}{r_{ij}} q_{ij}. \end{aligned} \tag{10.45}$$

The result (10.44) can be seen as the *discrete time extension of the Black and Scholes formula* given the environment:

$$J_0 = i,...,J_n = j, S(0) = S_0 . \tag{10.46}$$

2) *result with knowledge of* $J_0 = i$

If we only know the initial state of the environment $J_0 = i$, it is clear that the value of the call is given by

$$C_i(S_0,n) = \sum_{j=1}^{m} p_{ij}^{(n)} C_{ij}(S_0,n) \tag{10.47}$$

where, of course:

$$\left[p_{ij}^{(n)}\right] = \mathbf{P}^n. \tag{10.48}$$

3) *result with knowledge of* $J_n = j$

Proceeding as in the preceding section, the use of Bayes formula gives the following result now on *n* periods instead of one:

$$P\left(J_0=i\middle|J_n=j\right)=\frac{P\left(J_0=i,J_n=j\right)}{P\left(J_n=j\right)}$$
$$=\frac{a_i p_{ij}^{(n)}}{\sum_{k=0}^{m} a_k p_{kj}^{(n)}} \tag{10.49}$$

and so the value of the call given $J_n=j$, represented by $C^j(S_0,n)$, is given by:

$$C^j(S_0,n)=\sum_{i=1}^{m}\frac{a_i p_{ij}^{(n)}}{\sum_{k=0}^{m} a_k p_{kj}^{(n)}}C_{ij}(S_0,n). \tag{10.50}$$

4) *result with no environment knowledge*
Finally if we have no knowledge of the initial environment state but just its probability distribution given by (10.1), the value of the call denoted $C(S_0,n)$ is given by

$$C(S_0,n)=\sum_{i=1}^{m}a_iC_i(S_0,n) \tag{10.51}$$

or by

$$C(S_0,n)=\sum_{j=1}^{m}\sum_{k=1}^{m}a_k p_{kj}^{(n)}C^j(S_0,n). \tag{10.52}$$

10.3 The Multi-Period Discrete Markov Chain Limit Model

To construct our continuous time model on the time interval [0,*t*], *t* being the maturity time of the considered option, let us begin to consider a multi-period discrete Markov chain model with *n* periods where each period has length *h* so that we have equidistant observations at time 0,*h*,2*h*,...,*nh* with $n=\lfloor t/h\rfloor$.

We also suppose that in the approximated discrete time model, the environment process is now a homogeneous ergodic Markov chain defined by relations (10.1) and (10.2) and that (see Cox and Rubinstein (1985),p. 200) or relations (3.8), section 3.1 of this chapter), for each *n*, given $J_0,...,J_n,S(0)=S_0$ with $J_0=i,J_n=j$, we select, in each subinterval $\left[kh,(k+1)h\right]$, the following up and down parameters:

$$u_{j_k j_{k+1}} = e^{\sigma_{ij}\sqrt{\frac{t}{n}}}, d_{j_k j_{k+1}} = e^{-\sigma_{ij}\sqrt{\frac{t}{n}}},$$
$$q_{j_k j_{k+1}} = \frac{1}{2} + \frac{1}{2}\frac{\mu_{ij}}{\sigma_{ij}}\sqrt{\frac{t}{n}}, \tag{10.53}$$

depending thus on the two $m \times m$ non-negative matrices:

$$\left[\mu_{ij}\right], \left[\sigma_{ij}\right]. \tag{10.54}$$

From relations (10.40) and (10.41), it follows that, for all n:

$$E\left(\ln\frac{S(n)}{S_0} | J_0 = j_0, ..., J_n = j_n, S(0) = S_0\right) = \mu_{ij} t, \tag{10.55}$$

$$\mathrm{var}\left(\ln\frac{S(n)}{S_0} | J_0 = j_0, ..., J_n = j_n, S(0) = S_0\right) = \sigma_{ij}^2 t. \tag{10.56}$$

As our conditioning implies that we can follow the reasoning of Cox and Rubinstein (1985), we know that, for $n \to +\infty$:

$$\ln\frac{S(t)}{S_0} \prec N(\mu_{ij} t, \sigma_{ij}^2 t), \tag{10.57}$$

where $j_0 = i$ as the initial environment state observed at $t=0$ and j the environment state at time t.
Concerning the non-risky interest rates, we also suppose that, for all i and j, there exists $v_{ij} > 1$ such that the new return rate for all the periods $\left(kh, (k+1)h\right)$, denoted $\hat{r}_{ij}$, for $n \to +\infty$, satisfies the following condition:

$$(1 + r_{ij})^n \to (1 + \hat{r}_{ij})^t. \tag{10.58}$$

Now let $C_{ij}(S_0, n)$ represent the value at time 0 of a European call option with maturity n and exercise price K.
Using the proof of the Black and Scholes formula given by Cox and Rubinstein ((1985), pp. 205-208) but here with our parameters depending on all of the environment states i and j, we get under the conditions (10.53) and (10.58), for fixed t:

$$C_{ij}(S_0, n) \to C_{ij}(S_0, t) \tag{10.59}$$

where:

$$C_{ij}\ (S_0, t) = S_0 \Phi(d_{ij_t,1}) - K r_{ij}^{-t} \Phi(d_{ij_t,2}),$$
$$d_{ij_t,1} = \frac{\ln\frac{S_0}{K r_{ij}^{-1}}}{\sigma_{ij}\sqrt{t}} + \frac{1}{2}\sigma_{ij}\sqrt{t}, \tag{10.60}$$
$$d_{ij_t,2} = d_{ij,1} - \sigma_{ij}\sqrt{t}.$$

This result gives the value of the call at time 0 with *i* as initial environment state and *j* as environment state observed at time *t*, represented from now on by J_t.
If we want to use the classical notation in the Black and Scholes (1973) framework, we can define the instantaneous interest rate intensity ρ_{ij} such that:

$$r_{ij} = e^{\rho_{ij}} \tag{10.61}$$

so that the preceding formula (10.60) becomes now:

$$\begin{aligned} C_{ij_t}(S_0,t) &= S_0\Phi(d_{ij_t,1}) - Ke^{-\rho_{ij}t}\Phi(d_{ij_t,2}), \\ d_{ij_t,1} &= \frac{1}{\sigma_{ij}\sqrt{t}}\left(\ln\frac{S}{K} + \left(\rho_{ij} - \frac{\sigma_{ij}^2}{2}\right)t\right), \\ d_{ij_t,2} &= d_{ij,1} - \sigma_{ij}\sqrt{t}. \end{aligned} \tag{10.62}$$

10.4 The Extension Of The Black-Scholes Pricing Formula With Markov Environnment: The Janssen-Manca Formula

The last result (10.62) gives a first extension of the Black and Scholes formula in continuous time from the knowledge of the initial and final environment states, respectively J_0, J_t where J_t represents, as said above, the state of the environment at time *t*.
Now, always with the assumption that the Markov chain with matrix **P** is ergodic, we can extend results (10.43), (10.47) (10.50) and (10.52) valid for our discrete multi-period model to our continuous time model thus giving the following main result.

Proposition 10.1 (*Janssen and Manca* (*1999*))
Under the assumption that the Markov chain of matrix **P** *of the environment process is ergodic and that given the initial environment state* $i \in I$ *and the environment state at time t is* $j \in I$, *the non-risky rate is given by* ρ_{ij} *and the annual volatility by* σ_{ij}, *then we have the following results concerning the European call price at time* 0 *with exercise price K and maturity t:*
(1) *with knowledge of state* $J_0 = i, J_t = j$, *the call value is given by result* (10.62),
(2) *with knowledge of state* $J_0 = i$, *the call value represented by* $C_i(S_0,t)$ *is given by:*

$$C_i(S_0,t) = \sum_{j=1}^{m} \pi_j C_{ij}(S_0,t), \tag{10.63}$$

(3) *with knowledge of state* $J_t = j$, *the call value represented by* $C^j(S_0,t)$ *is given by:*

$$C^j(S_0,t) = \sum_{i=1}^{m} a_i C_{ij}(S_0,t), \tag{10.64}$$

(4) *without any environment knowledge, the call value represented by* $C(S_0,t)$ *is given by:*

$$C(S_0,t) = \sum_{i=1}^{m} a_i C_i(S_0,t) \tag{10.65}$$

or

$$C(S_0,t) = \sum_{j=1}^{m} \pi_j C^j(S_0,t). \tag{10.66}$$

Proof Result (1) is proved above.
Result (2) follows from relation (10.47) letting n go to $+\infty$ and then using result (1) and the assumption of ergodicity on the environment matrix chain $\mathbf{P}$.
Result (3) can easily be deduced from result (2) and relation (10.50).
Finally, result (4) follows immediately from relations (10.51) or (10.52) and results (2) and (3). □

Example
The **Example 10.1** is now treated in **Table 10.2** with the following annual volatility matrix

$$\boldsymbol{\sigma} = \begin{bmatrix} 0.20 & 0.30 \\ 0.25 & 0.35 \end{bmatrix}$$

Using for the matrix **P** being given by relation (10.33), using relation 2(9.77), the asymptotic distribution is given by:

$$\boldsymbol{\pi} = [\pi_1, \pi_2] = [0.977742, 0.032258]$$

Example 10.1				
K	80		K	95
S	100		S	100
0 to 0			0 to 0	
	1-t	Cij(100,t)	1-t	Cij(100,t)
	0.25	20.45	0.25	7.23
	0.5	21.10	0.5	8.98
	0.75	21.84	0.75	10.42
	1	22.61	1	11.67
1 to 0	0.25	20.57	0.25	8.08

	0.5	21.52	0.5	10.24
	0.75	22.56	0.75	11.98
	1	23.59	1	13.49
0 to 1	0.25	20.79	0.25	8.97
	0.5	22.13	0.5	11.54
	0.75	23.48	0.75	13.57
	1	24.76	1	15.33
1 to 1	0.25	21.12	0.25	9.88
	0.5	22.87	0.5	12.85
	0.75	24.53	0.75	15.18
	1	26.07	1	17.18
? To 1	0.25	20.8065	0.25	9.0155
	0.5	22.167	0.5	11.6055
	0.75	23.5325	0.75	13.6505
	1	24.8255	1	15.4225
? To 0	0.25	20.456	0.25	7.2725
	0.5	21.121	0.5	9.043
	0.75	21.876	0.75	10.498
	1	22.659	1	11.761
? To ?	0.25	20.46731	0.25	7.328726
	0.5	21.15474	0.5	9.125661
	0.75	21.92944	0.75	10.59969
	1	22.72889	1	11.87911

Table 10.2

In conclusion, the Janssen-Manca approach gives for the first time a new family of Black and Scholes formulae taking into account the economic and social environment showing that:

- a “good” extension of the classical Cox-Rubinstein model is possible,

-the model also extends the Black and Scholes model

-numerical results are possible.

Moreover, as the JM formulas are linear combinations of the classical BS results, the Greek parameters can also be computed and will be linear combinations of the Greek parameters given in section 6 and similarly for hedging coefficients.

We also add that, in our point of view, one of the main potential applications of our new model concerns the possibility to get a new way of acting with the Black

and Scholes formula with information related to the economic, financial and even political environment provided it can be modelled by an ergodic homogeneous Markov chain.
This model also gives the possibility of taking into account *anticipations* made by the investors in such a way as to incorporate them in their own option pricing and can also be used for models with financial crashes as well as to construct scenarios and particularly in the case of stress in a VaR type approach.

11 THE EXTENSION OF THE BLACK-SCHOLES PRICING FORMULA WITH A MARKOV ENVIRONMENT: THE SEMI-MARKOVIAN JANSSEN-MANCA-VOLPE FORMULA

11.1 Introduction

In this section, we present another semi-Markov extension of the Black and Scholes formula to the so-called Janssen-Manca-Volpe model to eliminate one of the restrictions of the Black and Scholes model that is the assumption of constant volatility upon time.
If there have been a lot of attempts to slacken this condition, as for example in the model of Hull and White (1985) where the concept of stochastic volatility is introduced, nevertheless, to our knowledge, in practice, no generalised model really supplants the classical Black and Scholes model.
Whilst comparing with the Markovian Janssen-Manca model of the preceding section, we develop another type of model. More precisely we present new semi-Markov models for the evolution of the volatility of the underlying asset.
In fact, the SM model presented here supposes a type of SM evolution for the volatility of an initial Black-Scholes model presented for the first time in an oral communication in the ETH Zurich (1995) by J. Janssen and in a different approach by E. Çinlar in an oral communication at the First Euro-Japanese meeting on Insurance, Finance and Reliability, held in Brussels in 1998, and leading to a generalization of the classical Black and Scholes formula for the pricing of European calls with easy numerical applications.

11.2 The Janssen-Manca-Çinlar Model

Hereby, we present our initial model of 1995 close to the oral presentation of Çinlar but he gives the formula for the pricing of a call option using the Markov renewal theory.

11.2.1 The JMC (Janssen-Manca-Çinlar) Semi-Markov Model (1995, 1998)

Let us consider a *two-dimensional* positive (*J-X*) process of kernel **Q** with as state space:

$$I=\{1,...,m\}. \tag{11.1}$$

This means that on the probability space $(\Omega,\Im,\mathrm{P})$, we define the *three-dimensional* process

$$\left((J_n,(X_n,\sigma_n)),n\geq 0\right) \tag{11.2}$$

with:

$$J_n\in I,(X_n,\sigma_n)\in\mathbb{R}^+\times\mathbb{R}^+, \tag{11.3}$$

such that:

$$\begin{aligned}&P\left(X_n\leq x,\sigma_n\leq\sigma,J_n=j\middle|(J_k,(X_k,\sigma_k)),k=0.1.....n-1\right)\\&=Q_{J_{n-1}j}(x,\sigma),p.s.\end{aligned} \tag{11.4}$$

We know that the $Q_{ij},i,j\in I$ can be written in the following form:

$$Q_{ij}(x,\sigma)=p_{ij}F_{ij}(x,\sigma) \tag{11.5}$$

where:

$$p_{ij}=P\left(J_n=j\middle|J_k,k\leq n-1,J_{n-1}=i\right), \tag{11.6}$$

$$F_{ij}(x,\sigma)=P\left(X_n\leq x,\sigma_n\leq\sigma\middle|(J_k,(X_k,\sigma_k)),k\leq n-1,J_{n-1}=i\right). \tag{11.7}$$

We also introduce the following r.v.:

$$\begin{aligned}&T_n=X_1+\cdots+X_n,n\geq 0,\\&N(t)=\sup\{n:T_n\leq t\},t\geq 0,\\&Z(t)=J_{N(t)},t\geq 0.\end{aligned} \tag{11.8}$$

As usual, the transition probability for the process $Z=\left(Z(t),T\geq 0\right)$ is designed by:

$$\phi_{ij}(t)=P\left(Z(t)=j\middle|Z(t)=i\right) \tag{11.9}$$

and the stochastic processes $(N(t),t\in\mathbb{R}^+),(Z(t),t\in\mathbb{R}^+)$ are respectively the Markov renewal counting and the semi-Markov processes.

To give the financial interpretation of our model, let us define on the probability space $(\Omega,\Im,P)$, the following filtration $\Im=(\Im_t,t\in\mathbb{R}^+)$,

$$\Im_t=\sigma((J_n,(X_n,\sigma_n)),n\leq N(t)). \tag{11.10}$$

Given $\Im_t$, let us consider the random time interval $\left[T_{N(t)},T_{N(t)+1}\right]$ on which we define the new stochastic process $(S(t),t\in\mathbb{R}^+)$, representing the value of the considered financial asset, as the solution of the stochastic differential equation:

$$\frac{dS}{S(t')} = \mu_{J_{N(t)}J_{N(t)+1}} dt' + \sigma_{J_{N(t)}J_{N(t)+1}} dW_{J_{N(t)}J_{N(t)+1}}(t' - T_{N(t)}), t' \in \left[T_{N(t)}, T_{N(t)+1}\right],$$
$$S(T_{N(t)}) = S(T_{N(t)}-), \tag{11.11}$$

where the process $(W_{J_{N(t)}J_{N(t)+1}}(t'), t' \geq 0)$ is a standard Brownian motion on $\left[T_{N(t)}, T_{N(t)+1}\right]$ defined on the basic probability space stochastically independent of $\left(J_{N(t)}, X_{N(t)}\right)$.

This model has the following financial meaning: at t=0, the asset starts from the known initial value S_0, the known initial j-state J_0 representing the state of the initial economic and financial environment. On the time interval X_1, the asset has the random volatility σ_1 and has as stochastic dynamics the SDE (11.11) with t=0; at time X_1, the J process has a transition to state J_1 and on the time interval $\left[T_1, T_2\right)$, the asset has the random volatility σ_2 and has as stochastic dynamics the SDE (11.11) with $N(t)$=1 and so on....

We always define $X_0 = 0, a.s.$

So, it is now clear that we have in fact a perturbed Black and Scholes model due to this random change of volatility; note that this model is quite general as, in fact, we have a random volatility on each time interval $\left[T_{N(t)}, T_{N(t)+1}\right]$.

Of course for m=1, we recover the classical Black-Scholes-Samuelson model for the description of an asset.

11.2.2 The explicit expression of S(t)

Given $J_{N(t)}, J_{N(t)+1}$, the Itô calculus gives the solution of the SDE (11.11) :

$$S(t') = S_{N(t)} e^{\left(\mu_{J_{N(t)}J_{N(t)+1}} - \frac{\sigma^2_{J_{N(t)}J_{N(t)+1}}}{2}\right)t'} e^{\sigma_{J_{N(t)}J_{N(t)+1}} W(t' - T_{N(t)})}, \tag{11.12}$$
$$t' \in \left[T_{N(t)}, T_{N(t)+1}\right].$$

Starting from the state S_0 at time 0 and given a scenario for the economic and financial environment $(J_0, J_1, ..., J_n, ...)$, this expression gives the explicit form of the trajectories of the process $(S(t), t \geq 0)$.

Now, given $(J_0, X_0, J_1, X_1, ..., J_{N(t)}, X_{N(t)}, J_{N(t)+1}, X_{N(t)+1})$, from relation (11.12), we get:

$$\ln\frac{S(t')}{S_{N(t)}}=\left(\mu_{J_{N(t)}J_{N(t)+1}}-\frac{\sigma^2_{J_{N(t)}J_{N(t)+1}}}{2})t'+\sigma_{J_{N(t)}J_{N(t)+1}}W\left(t'-T_{N(t)}\right)\right), \tag{11.13}$$
$$t'\in\left[T_{N(t)},T_{N(t)+1}\right],$$

so that for $t'\in\left[T_{N(t)},T_{N(t)+1}\right]$:

$$\ln\frac{S(t')}{S_{N(t)}}\prec N\left(\mu_{J_{N(t)}J_{N(t)+1}}-\frac{\sigma^2_{J_{N(t)}J_{N(t)+1}}}{2}\right)(t'-T_{N(t)}), \tag{11.14}$$
$$\sigma^2_{J_{N(t)}J_{N(t)+1}}(t'-T_{N(t)}),$$

$$E\left(\frac{S(t)}{S_{N(t)}}\middle|\Im_t,J_{N(t)+1}\right)=e^{\mu_{J_{N(t)},J_{N(t)+1}}(t'-T_{N(t)})}, \tag{11.15}$$

$$\operatorname{var}\left(\frac{S(t)}{S_{N(t)}}\middle|\Im_t,J_{N(t)+1}\right)=e^{2\mu_{J_{N(t)},J_{N(t)+1}}(t'-T_{N(t)})}\left(e^{\sigma^2_{J_{N(t)}J_{N(t)+1}}(t'-T_{N(t)})}-1\right). \tag{11.16}$$

Let us suppose that the random variables $S_0,J_0,X_1,J_1,...,J_{N(t)},X_{N(t)+1},J_{N(t)+1}$ are given; it follows that the conditional distribution function of $\frac{S(t)}{S_0}$ is a log-normal distribution, i.e.:

$$\ln\frac{S(t)}{S_0}\prec$$
$$N\left(\mu_{J_0J_1}X_1+\cdots+\mu_{J_{N(t)}J_{N(t)+1}}(t-T_{N(t)}),\sigma^2_{J_0J_1}X_1+\cdots+\sigma^2_{J_{N(t)}J_{N(t)+1}}(t-T_{N(t)})\right). \tag{11.17}$$

11.3 Call Option Pricing

Now to get a useful model, let us proceed as in Janssen and Manca (1999); for a fixed *t*, we assume that all the parameters μ,σ only depend on $J_0,J_{N(t)},J_{N(t)+1}$ and *t* is represented by

$$\mu_{J_0J_{N(t)}J_{N(t)+1}},\sigma_{J_0J_{N(t)}J_{N(t)+1}} \tag{11.18}$$

so that from relation (11.17):

$$\ln\frac{S(t)}{S_0}\prec N\left(\left(\mu_{J_0J_{N(t)}J_{N(t)+1}}-\frac{1}{2}\sigma^2_{J_0J_{N(t)}J_{N(t)+1}}\right)t,\sigma^2_{J_0J_{N(t)}J_{N(t)+1}}t\right). \tag{11.19}$$

Of course, we can always simplify our basic assumption by suppressing the dependence with respect to $J_{N(t)+1}$ and even to $J_{N(t)}$.

Nevertheless, we think that the dependence from the future environment state $J_{N(t)+1}$ is quite important as it gives the possibility for the first time to model the stochastic asset evolution taking into account this anticipation of the next future state.

Let us now consider a European call option with *t* as maturity time, *K* as exercise price that we must price at time 0.

If we want to assume that there is no arbitrage possibility, we must impose that

$$\mu_{J_0 J_{N(t)} J_{N(t)+1}} = \delta_{J_0 J_{N(t)} J_{N(t)+1}} \tag{11.20}$$

where $\delta_{J_0 J_{N(t)} J_{N(t)+1}}$ represents the equivalent instantaneous non-risky return on $[0,t]$ given $J_0, J_{N(t)}, J_{N(t)+1}$. Doing so, we will use the risk neutral measure under which the forward value of the asset is a martingale; otherwise we work with the initial "physical" measure more appropriate for insurance than for finance.

Knowing $J_0, J_{N(t)}, J_{N(t)+1}$ and working with the risk neutral measure, we can compute the value of the call at time 0 using the classical Black and Scholes formula:

$$\begin{aligned}
&C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0,t) = S_0 \Phi(d_{J_0 J_{N(t)} J_{N(t)+1},1}) - K r_{J_0 J_{N(t)} J_{N(t)+1}}^{-t} \Phi(d_{J_0 J_{N(t)} J_{N(t)+1},2}),\\
&d_{J_0 J_{N(t)} J_{N(t)+1},1} = \frac{\ln \dfrac{S_0}{K r_{J_0 J_{N(t)} J_{N(t)+1}}^{-1}}}{\sigma_{J_0 J_{N(t)} J_{N(t)+1}} \sqrt{t}} + \frac{1}{2} \sigma_{J_0 J_{N(t)} J_{N(t)+1}} \sqrt{t},\\
&d_{J_0 J_{N(t)} J_{N(t)+1},2} = d_{J_0 J_{N(t)} J_{N(t)+1},1} - \sigma_{J_0 J_{N(t)} J_{N(t)+1}} \sqrt{t},\\
&v_{J_0 J_{N(t)} J_{N(t)+1}} = e^{\delta_{J_0 J_{N(t)} J_{N(t)+1}}}.
\end{aligned} \tag{11.21}$$

To get the formula of the call only knowing S_0, J_0, we must use the following formula:

$$C_{J_0}(t) = E\left(C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0,t) \middle| J_0, S_0\right). \tag{11.22}$$

From the theory of semi-Markov processes, we get:

$$\begin{aligned}
&C_{J_0}(t) = E\left(C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0,t) \middle| J_0, S_0\right),\\
&C_{J_0}(t) = \sum_{j\in I} \sum_{k\in I} P_{J_0 j}(t) p_{jk} C_{J_0 jk}(S_0,t).
\end{aligned} \tag{11.23}$$

If we have no information about the initial state J_0, we get of course the following formula:

$$\begin{aligned}
&C(t) = E\left(C_{J_0}(t)\right) = E\left(E\left(C_{J_0 J_{N(t)} J_{N(t)+1}}(S_0,t) \middle| J_0, S_0\right)\right),\\
&C(t) = \sum_{i\in I} a_i C_i(t).
\end{aligned} \tag{11.24}$$

Remark 11.1 Numerical treatments are possible

11.4 Stationary Option Pricing Formula

In option pricing, it is nonsense to let t tend towards $+\infty$; nevertheless, we can use the limit reasoning proposed by Janssen by supposing that on the time horizon $[0,t]$, the semi-Markov environment has more and more transitions in this finite time period.
We can model this situation under the assumption that the conditional sojourn time means $b_{ij}, i, j \in I$ satisfy the conditions

$$\begin{aligned} & b_{ij} = \varepsilon\varsigma_{ij,} \ \varepsilon > 0, \\ & b_{ij} = E\left(X_n \left| J_{n-1} = i, J_n = j\right.\right) \end{aligned} \tag{11.25}$$

so that:

$$\begin{aligned} & \eta_i = \sum_{j\in I} p_{ij} b_{ij} = \varepsilon \sum_{j\in I} p_{ij}\varsigma_{ij} = \varepsilon\theta_i, i \in I, \\ & \theta_i = \sum_{j\in I} p_{ij}\varsigma_{ij}. \end{aligned} \tag{11.26}$$

From the asymptotic theory of semi-Markov processes, we know that:

$$\lim_{\varepsilon\to 0} P\left(J_{N(t)} = j, J_{N(t)+1} = k\right) = \frac{\pi_i p_{jk}\varsigma_{jk}}{\sum_{l=1}^{m}\pi_l\theta_l}, i, j \in I, \tag{11.27}$$

where the vector $\left(\pi_1, ..., \pi_m\right)$ is the unique stationary distribution of the embedded Markov chain of matrix **P** supposed to be ergodic.
The new parameters ς_{jk}, $i, j, k \in I$ represent factors expressing the proportionality of the sojourn in each environment state.
Now the result (11.23) becomes:

$$C_{J_0}(t) = \sum_{j\in I}\sum_{k\in I} \frac{\pi_j p_{jk}\varsigma_{jk}}{\sum_{l=1}^{m}\pi_l\theta_l} C_{J_0 jk}(S_0, t). \tag{11.28}$$

From (11.24), we get

$$C(t) = \sum_{i\in I} a_i \sum_{j\in I}\sum_{k\in I} \frac{\pi_j p_{jk}\varsigma_{jk}}{\sum_{l=1}^{m}\pi_l\theta_l} C_{ijk}(S_0, t). \tag{11.29}$$

This last formula replaces the Black and Scholes formula without any a priori information at time 0 except of course the initial value of the asset S_0.

In conclusion, the new model proposed here extends the classical Black and Scholes formula in the case of the existence of an economic and financial environment modelled with a homogeneous semi-Markov process taking into

account this environment not only at the time of pricing but also before and after the maturity date.
This new family of Black and Scholes formulae seems to be more adapted to the reality, particularly when taking into account the anticipations of the investor or the consideration of stress scenario in the philosophy of the VaR approach.

12 MARKOV AND SEMI-MARKOV OPTION PRICING MODELS WITH ARBITRAGE POSSIBILITY

The aim of this last part is the presentation of new models for option pricing, discrete in time and within the framework of Markov and semi-Markov processes as an alternative to the classical Cox-Rubinstein model and giving arbitrage possibilities. They were introduced by Janssen, Manca and Di Biase (1998). Both cases of European and American options are considered and possible extensions are given.

12.1 Introduction to the Janssen-Manca-Di Biase models

Let us consider an asset observed on a discrete time scale

$$\left\{0,1,...,t,...T\right\}, T<\infty \tag{12.1}$$

having $S(t)$ as market value at time t. To model the basic stochastic process

$$(S(t),\ t=0,1,...,T), \tag{12.2}$$

we suppose that the asset has known minimal and maximal values so that the set of all possible values is the closed interval $[S_{\min}, S_{\max}]$ partitioned in a subset of m subclasses.
For example, if S_0 is the value of the asset at time 0, we can put:

$$\begin{aligned} S_0 &= \frac{S_{\max}-S_{\min}}{2}, \\ S_k &= S_0 + k\Delta, k=1,...,\upsilon, \\ S_{-k} &= S_0 - k\Delta, k=1,...,\upsilon, \\ \Delta &= \frac{S_{\max}-S_{\min}}{2\upsilon}, \end{aligned} \tag{12.3}$$

υ being arbitrarily chosen.
This implies that the total number of states is $2\upsilon+1$.In the sequel, we will order these states in the natural increasing order and use the following notation for the state space:

$$I=\{-\upsilon,-(\upsilon-1),...,0,1,...,\upsilon\}. \tag{12.4}$$

We can also introduce different step lengths following up or down movements and so consider respectively Δ, Δ'.
It is also possible to let

$$S_{\max} \to +\infty \tag{12.5}$$

and

$$T \to +\infty \tag{12.6}$$

particularly to get good approximation results.

Let us suppose we want to study a call option of maturity T and exercise price $K = k_0\Delta$ both in European and American cases bought at time 0.

So, in the European case, the intrinsic value of the option is given by:

$$C(T) = \max\{0, S(T) - K\}. \tag{12.7}$$

For the American case, the optimal time for exercising is given by the random time τ such that:

$$\max_{t=1,\dots,T} \max\{0, S_t - K\} = \max\{0, S_\tau - K). \tag{12.8}$$

To get results, we must now introduce in the following section a stochastic model for the S-process.

12.2 The Homogeneous Markov JMD (Janssen-Manca-Di Biase) Model For The Underlying Asset

Let us suppose that we are working on the filtered probability space $(\Omega, \Im, (\Im_t)P)$.

In our first model, we will suppose that the underlying asset S is a homogeneous Markov chain with matrix:

$$\mathbf{P} = \left[p_{ij}\right] \tag{12.9}$$

on the state space I given by relation (12.4).

It follows that, at time t, given the knowledge of the asset value $S(t) = S_t$ the market value of the option at time t, $C(t)$, thus with a remaining maturity T-t and exercise price K given by $K = k_0\Delta,$ has as probability distribution:

$$\begin{aligned} &P(C(T) = (j - k_0)\Delta) = p_{S,j}^{(T-t)}, j > k_0, \\ &P(C(T) = 0) = \sum_{l \le k_0} p_{S,j}^{(T-t)}. \end{aligned} \tag{12.10}$$

This result gives the possibility to compute all interesting parameters concerning C. For example, the mean of $C(t)$ has the value:

$$E(C(T) \big| S(t) = S_t) = \sum_{l > k_0} p_{S,j}^{(T-t)} (l - k_0)\Delta. \tag{12.11}$$

Of course, we have to compute the present value at time t with the non-risky unit period interest rate r so that the value of the call at time t is given by:

$$C(t) = v^{T-t} E(C(T) \big| S(t) = S_t) = v^{T-t} \sum_{l>k_0} p_{S,j}^{(T-t)} (l-k_0)\Delta,$$
$$v = \frac{1}{1+r}. \tag{12.12}$$

If the matrix **P** is ergodic, then if $T-t$ is large enough, results (12.10) and (12.11) can be well approximated by:

$$\begin{aligned} &P(C(T) = (j-k_0)\Delta) = \pi_j, j > k_0, \\ &P(C(T) = 0) = \sum_{l \le k_0} \pi_l, j \le k_0, \\ &E(C(T) \big| S(t) = S_t) = \sum_{l>k_0} \pi_j (l-k_0)\Delta, \\ &C(t) = v^{T-t} \sum_{l>k_0} \pi_j (l-k_0)\Delta. \end{aligned} \tag{12.13}$$

Of course the vector

$$\pi = (\pi_{-v}, ..., \pi_0, ..., \pi_v) \tag{12.14}$$

is the steady-state vector related to the matrix **P**.

12.3 Particular Cases

As we said in our introduction, our homogeneous Markov model contains as a very special case the famous CRR binomial model but with fixed minimal and maximal values. It suffices to select a Markov matrix **P** with the structure

$$\begin{bmatrix} * & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & * \end{bmatrix} \tag{12.15}$$

and as the Cox-Rubinstein model has a multiplicative form, we can consider that:

$$\Delta = \begin{cases} (u-1)S_0, u > 1, S > S_0, \\ (1-d)S_0, d < 1, S < S_0. \end{cases} \tag{12.16}$$

Remark 12.1 Under (12.5), the matrix **P** has an infinite number of rows and columns.

We can also get the **trinomial model** if we put in (12.15) a non-zero main diagonal and so on.

12.4 Numerical Example For The JMD Markov Model

To illustrate numerically our first model, let us suppose that we are interested in an asset whose possible values are restricted to the following ones;

maximum value: state 3=1650,
intermediary values: state 2=1600, state 1=1550, state 0=1500,
state -1=1450, state -2=1400,
minimum value: state -3=1350.

With the used notation, this means that $S_0 = 1500,\ \Delta = 50.$ Moreover, we also suppose that the transition matrix **P**, with as unit step the week, is given by

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{7} & \frac{2}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{7} & \frac{3}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} & \frac{2}{7} & \frac{1}{7} & \frac{1}{7} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{bmatrix}. \tag{12.17}$$

It is easily seen that the matrix **P** is ergodic with as unique stationary distribution: (0.10002, 0.13336, 0.27228, 0.23737, 0.16927, 0.07539, 0.01231).
Then starting at time 0 in state 1500 with a maturity time of 16 weeks, the asymptotic value of the European call option expectation with 1500 as exercise price is 41.95 and the call value at time 0 is 41.328.
Table 12.1 gives option expectations and option values with different exercise prices:

Exercise price	Option expectation	Option value
1350	174.106	171.512
1400	124.721	122.826
1450	79.1059	77.927
1500	41.9538	41.328
1550	16.6704	16.422
1600	5.00113	4.927
1650	0	0

Table 12.1: Markov option computation

Let us now consider the transient behaviour, meaning that we will consider the maturity as a parameter expressed in *n* weeks. **Table 12.2.1**, gives option expectations, Table **12.2.2** option values with as exercise price 1500 and for different maturity times from 1 to 16 weeks.

	STATE						
n	-3	-2	-1	0	1	2	3
1	75.00	75.00	57.14	25.00	14.29	7.14	0.00
2	60.71	53.57	46.93	38.39	30.10	20.41	16.96
3	50.02	48.40	43.39	40.60	37.08	31.61	31.39
4	45.70	44.92	42.79	41.11	39.61	37.39	37.44
5	43.70	43.30	42.35	41.57	40.84	39.87	39.81
6	42.76	42.58	42.13	41.78	41.45	40.98	40.96
7	42.33	42.24	42.04	41.87	41.72	41.50	41.50
8	42.13	42.09	41.99	41.92	41.84	41.75	41.74
9	42.03	42.02	41.97	41.94	41.90	41.86	41.86
10	41.99	41.98	41.96	41.95	41.93	41.91	41.91
11	41.97	41.97	41.96	41.95	41.94	41.93	41.93
12	41.96	41.96	41.96	41.95	41.95	41.94	41.94
13	41.96	41.96	41.95	41.95	41.95	41.95	41.95
14	41.96	41.96	41.95	41.95	41.95	41.95	41.95
15	41.95	41.95	41.95	41.95	41.95	41.95	41.95
16	41.95	41.95	41.95	41.95	41.95	41.95	41.95

Table 12.2.1: option expectation

	STATE						
n	-3	-2	-1	0	1	2	3
1	70.93	74.93	57.09	24.98	14.27	7.14	0.00
2	60.60	53.47	46.85	38.32	30.05	20.37	16.93
3	49.88	48.27	43.26	40.48	36.98	31.53	31.31

4	45.53	44.75	42.63	40.26	39.45	37.25	37.30
5	43.50	43.10	42.15	41.38	40.65	39.68	39.63
6	42.22	42.34	41.90	41.54	41.21	40.75	40.73
7	42.05	41.97	41.76	41.60	41.45	41.23	41.22
8	41.81	41.77	41.68	41.60	41.53	41.43	41.43
9	41.68	41.66	41.62	41.58	41.55	41.51	41.50
10	41.60	41.59	41.57	41.55	41.54	41.52	41.52
11	41.54	41.54	41.53	41.52	41.51	41.50	41.50
12	41.49	41.49	41.49	41.48	41.48	41.47	41.47
13	41.45	41.45	41.45	41.44	41.44	41.44	41.44
14	41.41	41.41	41.41	41.41	41.41	41.41	41.40
15	41.37	41.37	41.37	41.37	41.37	41.37	41.37
16	41.33	41.33	41.33	41.33	41.33	41.33	41.33

Table 12.2.2: option value

12.5 The Continuous Time Homogeneous Semi-Markov JMD Model For The Underlying Asset

With the generalisation of electronic trading systems, it seems more adapted to construct a time continuous model for which the changes in the values of the underlying process may depend on the time it remained unchanged before a transition.
Also, let

$$((S_n, T_n)\ n = 0,1,...) \tag{12.18}$$

be the successive states and time changes of the considered asset.
The Janssen-Manca semi-Markov continuous model without AOA starts from the basic assumption that the process (12.18) is a semi-Markov process of kernel **Q**.
It follows that, at time t in state $S(t)=S_t$, the market value of the considered European option with maturity $T-t$ has as probability distribution at maturity time

$$\begin{aligned} &P(C(T)=(j-k_0))=\phi_{S_t j}(T-t), j>k_0, \\ &P(C(T)=0)=\sum_{l\le K_0}\phi_{S_t j}(T-t), j\le k_0. \end{aligned} \tag{12.19}$$

Of course, the matrix $\mathbf{\Phi}(t)$ represents the transition probabilities for the considered semi-Markov process (see Chapter 3, relation (10.2))
This result gives the possibility to compute all interesting parameters concerning C. For example, the mean of $C(T)$ has the value:

$$E(C(T)=|S(t)=S_t)=\sum_{j>k_0}\phi_{S_t j}(T-t)(j-k_0)\Delta. \tag{12.20}$$

The pricing of the option at time t is here given by the conditional market value $C(t)$:

$$C(S_t,t)=v^{T-t}\sum_{j>k_0}\phi_{S_t j}(T-t)(j-k_0)\Delta \qquad (12.21)$$

which is the Janssen-Manca-Di Biase formula for the considered semi-Markov model.
If the semi-Markov process is ergodic, then, if $(T-t)$ is large enough, results (12.19) can be well approximated by:

$$\begin{aligned}&P(C(T)=(j-k_0))=\tilde{\pi}_j\,,\,j>k_0\,,\\&P(C(T)=0)=\sum_{l\le K_0}\tilde{\pi}_l\,,\,j\le k_0\,.\end{aligned} \qquad (12.22)$$

The *stationary* version of the Janssen-Manca-Di Biase formula is thus given by

$$C(S_t,t)=v^{T-t}\sum_{j>k_0}\tilde{\pi}_j\phi_{S_t j}(j-k_0)\Delta. \qquad (12.23)$$

Of course the vector $(\tilde{\pi}_1,...,\tilde{\pi}_m)$ is the asymptotic distribution of the embedded semi-Markov process.
The evaluation of assets, formally is continuous, but substantially is given in the discrete case; furthermore, facing the numerical solution of a continuous time semi-Markov process gives problems of numerical and stochastic convergence. For these reasons, we can face our problem with the discrete time homogeneous semi-Markov process as introduced in Chapter 4.

12.6 Numerical Example For The Semi-Markov JMD Model

We will only give a numerical example for the semi-Markov model in the asymptotic case, i.e., values of the option expectation and of the options for large maturities.
As data, we just need as supplementary information, the conditional mean sojourn times being computed by relations (1.15) of Chapter 4.The used values are given by the following matrix $\mathbf{\Sigma}$:

$$\mathbf{\Sigma}=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 & 2 & 1 & 1 & 1 \\ 1 & \frac{1}{4} & \frac{1}{4} & 1 & 2 & 1 & 1 \\ 2 & 1 & \frac{1}{2} & 1 & 2 & 2 & 1 \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 \\ 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 2 & 1 \\ 1 & 1 & 2 & 1 & \frac{1}{2} & \frac{1}{2} & 2 \\ 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}. \tag{12.24}$$

In this case, the asymptotic distribution for the semi-Markov process is: (0.09487, 0.12650, 0.38238, 0.15352, 0.15013, 0.08358, 0.00902).
Then starting at time 0 in state 1500, the asymptotic value of the call option expectation with 1500 as exercise price is 46 and the call value is 45.315.
The following table gives option expectations and option values with different exercise prices:

Exercise price	Option expectation	Option value
1350	178.78	176.119
1400	129.234	127.308
1450	83.8638	82.614
1500	46.0002	45.315
1550	15.8126	15.577
1600	4.74378	4.673
1650	0	0

Table 12.3: semi-Markov option computation

12.7 Conclusion

The JMD models presented here give a semi-Markov approach for the pricing of option financial products working in discrete time and with a finite number of possible values for the imbedded asset, which is always the case from the numerical point of view.
The main interest of these models is that they work even when there are possibilities of arbitrage, that is to say for the most frequent cases. Of course, one

of the main difficulties for applying this model is the fitting of the needed data and this is only of interest in case of asymmetric information so that the economic agent can believe in his own information, knowing that he will always be in a risky situation expecting gain but still worried about the possibility to lose as in the case of a real-life situation!

It is also important to point out that the numerical examples are coherent; nevertheless, they are significant differences according to the model used: Markov or semi-Markov, so that it is very important to select the most concrete one.

Another possibility is to use the more classical and less risky way of building likely scenarios for these data and to study their possible consequences.

Chapter 6

OTHER SEMI-MARKOV MODELS IN FINANCE AND INSURANCE

1 EXCHANGE OF DATED SUMS IN A STOCHASTIC HOMOGENEOUS ENVIRONMENT

1.1 Introduction

According to several authors, the theory of financial operations evaluation can be introduced from an axiomatic point of view (for a more complete treatment see Volpe di Prignano (1985)) and any financial problem can be dealt with from the axiomatic approach. The challenge is to find the mathematical relations that must be solved in order to find quantitative answers to financial questions.
Of course, the choice of the interest evolution law is crucial.
For example if we call:

$$\nu(t_1,t_2) \tag{1.1}$$

an exchange factor, where:

$$\nu(t_1,t_2)\begin{cases} >1 & if \quad t_1 < t_2, \\ =1 & if \quad t_1 = t_2, \\ <1 & if \quad t_1 > t_2, \end{cases} \tag{1.2}$$

it is not always true that:

$$\nu(t_1,t_2)\nu(t_2,t_3) = \nu(t_1,t_3) \tag{1.3}$$

and then it is not possible to use the compound interest law to describe the phenomenon which is the object of our study; we have to select another interest law.
Furthermore the problem of the construction of exchange factors is more useful in a stochastic environment, in the sense that the exchange factors can be considered as random variables and, to follow time evolution, we need a stochastic process approach.
In this chapter our aim is to show how, by means of SMP, it is possible to introduce a stochastic environment in the axiomatic approach to describe a lot of financial operations.
Furthermore we would like to show that the semi-Markov approach could be an alternative to the approach under the hypothesis of the absence of opportunity of arbitrage (AOA, see Chapter 5) which implies the use of the risk free measure for a asset evaluation.
The semi-Markov approach enables us to get financial evaluations by means of the physical measure. This fact is really important in the sphere of actuarial

science because, if it is possible to know or to make good estimates of claims by means of statistics, an insurance company can evaluate its premiums and limit its risk of ruin.
As we have already shown in section 12 of Chapter 5, in finance, these models can be seen as an alternative approach to the methods presented by Black and Scholes (1973) for option evaluation and for Vasicek (1977) for bond evaluation in continuous time evolution of a general financial problem.
In insurance there are other approaches that are really useful, see Bühlmann (1992), (1994), Norberg (1995), but these approaches do not really consider financial choice problems that will be developed in this chapter.

1.2 Deterministic Axiomatic Approach To Financial Choices

Fundamentally, mathematics of finance is based on the study of investor preferences between *dated sums*, given by a pair of real numbers i.e:

$$(S,t), S,t \in \mathbb{R}, \tag{1.4}$$

S representing a sum and t a time.

Definition 1.1 *The investor prefers the pair* (S_2,t_2) *to the pair* (S_1,t_1) *means that he prefers to get the sum* S_2 *at time* t_2 *instead of the sum* S_1 *at time* t_1*.*
In this case, the preference relation defined on the set of dated sums is represented by the following notation:

$$(S_1,t_1) \preceq (S_2,t_2). \tag{1.5}$$

Definition 1.2 *If*

$$(S_1,t_1) \preceq (S_2,t_2) \text{ and } (S_2,t_2) \npreceq (S_1,t_1), \tag{1.6}$$

then the preference is called strict and is thus represented as:

$$(S_1,t_1) \prec (S_2,t_2). \tag{1.7}$$

Definition 1.3 *If*

$$(S_1,t_1) \preceq (S_2,t_2) \text{ and } (S_2,t_2) \preceq (S_1,t_1), \tag{1.8}$$

the two dated sums are called indifferent and we write:

$$(S_1,t_1) \approx (S_2,t_2). \tag{1.9}$$

As given in Duffie (1988) the ideal should be that this indifference relation is an equivalence relation (reflexive, symmetric and transitive).
Furthermore, for economic and financial reasons, it is natural to assume that:

$$(S,t_1) \prec (S,t_2) \begin{cases} t_1 > t_2 & \text{if} \quad S > 0, \\ t_1 < t_2 & \text{if} \quad S < 0, \end{cases} \tag{1.10}$$

$$(S_1,t) \prec (S_2,t) \text{ iff } S_1 < S_2. \tag{1.11}$$

For the very particular case of a flat interest rate curve with a yearly interest rate r, Volpe di Prignano (1985), the strict preference relation defined by:

$$(S_1,t_1) \prec (S_2,t_2) \text{ iff } \frac{S_1}{(1+r)^{t_1}} < \frac{S_2}{(1+r)^{t_2}} \tag{1.12}$$

evaluated at time 0 is at the basis of all classical finance as it represents comparisons of present values computed at time 0.

For any yield curve in which r_t represents the interest rate for an investment made at time 0 and of maturity t, the preceding relation becomes:

$$(S_1,t_1) \prec (S_2,t_2) \text{ iff } \frac{S_1}{(1+r_{t_1})^{t_1}} < \frac{S_2}{(1+r_{t_2})^{t_2}}. \tag{1.13}$$

Let us remark that if the considered yield curve presents some local inversion phenomena so that the curve is no longer globally increasing, then relation (1.7) is no longer a total order relation and moreover properties (1.12) and (1.13) are not always verified.

In the case in which a total relation order is defined and relations (1.10) and (1.11) are satisfied, given an investor and a certain instant, we can define a so-called indifference relation $\Re$ on dated sums as follows:

$$(S_1,t_1)\Re(S_2,t_2) \Leftrightarrow (S_1,t_1) \approx (S_2,t_2)\ . \tag{1.14}$$

It is clear that this indifference relation can change with time depending on the evolution of the money utility function of the investor and the economic, political and financial environment under consideration and in the same way, under the same assumptions, two different investors usually have different indifference relations.

The presented static approach to the theory of the preference between dated sums can and must be extended from a dynamic point of view as follows: under our assumptions, given t_1, t_2 and S_1, there exists one and only one amount S_2 such that (1.14) holds and so we can define the following three-variable functions giving this value:

$$S_2 = \varphi(t_1,t_2,S_1). \tag{1.15}$$

It is clear that this is a continuous approach to the problem, all three variables being continuous. But for practical applications, it is possible to discretize the sums to obtain the set

$$I = \{S_1,S_2,...,S_n\},\ S_1 < S_2 < \cdots < S_m \tag{1.16}$$

as all possible amount values.

All remains unchanged but now relation (1.15) represents a three-variable function in which one of the variables and the function can only assume a finite number of values.

Though relation (1.15) represents a non-homogeneous time environment, we begin with the time homogeneous case.

In a deterministic static homogeneous case, the indifference relation (1.14) becomes:

$$(S_1,t_1) \approx (S_2,t_2) \Leftrightarrow (S_1,t_1+h) \approx (S_2,t_2+h) \ \forall h \in \mathbb{R}, \tag{1.17}$$

so that a time translation is indifferent for the choice; only the length of the financial operation, $(t_2 - t_1)$, assumes relevance.

In the dynamic approach, equation (1.15) becomes:

$$S_2 = \varphi(S_1,t). \tag{1.18}$$

In this case we have a two-variable function: the time variable represents the length of the operation and S_1 the initial sum.

1.3 The Homogeneous Stochastic Approach

Though usually an investor who wants to invest a given amount at a given time and for a given period cannot predict the final result of his investment, it seems reasonable to assume that he can know an interval $[S',S'']$,

$$S \in [S',S''], \tag{1.19}$$

within which his final return *S* must be situated and thus with a minimal value S',eventually 0, and a maximum one S''.

In this way, the final result *S* of his financial operation clearly can be seen as a random variable.

In the static approach, the maximal information of the investor will be the distribution function of this r.v.

If *p* represents the density of the r.v., then he can compute the mean and variance of his investment:

$$\overline{S} = \int_{S'}^{S''} S \cdot p(S)dS, \tag{1.20}$$

$$\sigma^2(S) = \int_{S'}^{S''} (S-\bar{S})^2 p(S)dS \tag{1.21}$$

and with this information (*expected value* and *risk measure*) at the basis of the Markovitz theory, the investor can decide to make the investment or not.

If the function *p* represents all the information to solve the static approach, it is much more complicated for the stochastic dynamic approach.

The stochastic model needs the introduction of a probability space $(\Omega,\Im,P)$ where *P* is the probability measure describing the stochastic dynamics of the process

$$S = (S(t), t = t_1,...,t_n). \tag{1.22}$$

In particular for each time *t*, it is also possible to get the mean and the variance of $S(t)$ but of course, we can do more with the construction of a realistic stochastic model.

The next dynamic stochastic model will be characterized by the following transition probability function:

$$p(S_1,S_2,t) = P(\,S(t) = S_2 \,|\, S(0) = S_1\,),$$
$$S_1,S_2 \in [S',S''], t \in [\,0,T\,]. \tag{1.23}$$

The assumption of homogeneity implies that:

$$P(\,S(s+\Delta t) = S_2 \,|\, S(s) = S_1\,) = p(S_1,S_2,\Delta t),$$
$$s, s+\Delta t \in [\,S',S''\,]. \tag{1.24}$$

It is clear that the knowledge of the function *p*(.,.,.) defined by (1.23) gives the possibility to compute the expected values and risks of financial operations.
In fact this model can be improved with the introduction of a two-dimensional process with time as a second component.

1.4 Continuous Time Models With Finite State Space

When we deal with the dynamic stochastic approach to a general financial problem we are defining a stochastic process describing the evolution of the investment during some period of time.
This evolution can be studied by means of an HSMP of kernel **Q;** in fact as already explained, one follows the evolution of the system states with a stochastic time length.
We denote the state space I by:

$$I = \{\,S_1,S_2,\ldots,S_m\,\},$$
$$S_1 = S_{\min} < S_2 < \cdots < S_m = S_{\max}. \tag{1.25}$$

We will try to give a financial meaning to all the variables that are involved in an SMP.
If J_n represents the value of the sum at transition *n*, it is usual to assume that the stochastic process

$$(J_n, n \ge 0) \tag{1.26}$$

is a homogeneous SMP where the transition matrix of the embedded MC is:

$$\mathbf{P} = [\,p_{ij}\,]. \tag{1.27}$$

Here, p_{ij} represents the probability of going from the state i to the state j, where in general the state *k* represents the sum value $S_k \in I$.
We also introduce the r.v. T_n representing the time at transition *n*, i.e., the time of the *n*th sum transition, where clearly:

$$0 \le T_0 \le T_1 \le \cdots \le T_n \le \cdots, \tag{1.28}$$

T_0 representing the time of the initial investment.
The basic assumption of the model considered here is that the (*J*-*T*) process is a continuous time homogeneous semi-Markov process of kernel **Q**.
Here, the general element $Q_{ij}(t)$ of the semi-Markov matrix kernel **Q** represents the probability that after a time *t* from the *n*-th transition, the value of the

financial operation is S_j, given that the value of the operation was S_i at time T_n and that the $(n+1)$ th transition happened in a time less than or equal to *t*.
$H_i(t)$ gives the probability that the financial operation value will change, given that the value at the moment of the T_n was S_i and that the value change will take place in a time less than or equal to *t*.
Another interesting r.v. is the sojourn time in one of the states after a transition in this state. For example, if at time T_n the r.v. sum gets the value S_i, the probability of this sum becoming S_j in a time length *t* will be represented, as we already know, by the increasing distribution function $F_{ij}(t)$.
The associated homogeneous semi-Markov process *Z* represents the value at time *t* of the dated sum.
$\phi_{ij}(t)$ represents the transition probabilities of the *Z*-process.
The evolution equations of the *Z*-process are reported to give their financial meaning:

$$\phi_{ij}(t) = \delta_{ij}\left(1 - H_i(t)\right) + \sum_{k \in E} \int_0^t \dot{Q}_{ik}(s)\phi_{kj}(t-s)ds\,. \tag{1.29}$$

$\phi_{ij}(t)$ represents the probability that the sum S_i at time 0 will be S_j after a time *t*.
$\delta_{ij}\left(1 - S_i(t)\right)$ represents the probability that the value of the investment will remain S_i after a time *t*; this element has sense iff *i=j*.
$\sum_{k \in E} \int_0^t \dot{Q}_{ik}(s)\phi_{kj}(t-s)ds$ represents the probability that the value of the financial operation will be S_j after a time *t*, taking into account that at initial time the value was S_i and that the financial operation changed value at least once.
Instead of working with continuous time, we can also consider discrete time; this generally implies a simplification for the numerical treatment as we have already explained before.

1.5 Discrete Time Model With Finite State Space

We will now look at an application of the model defined in the previous paragraph in real problems.
As we noted previously, in the real world, we are not usually interested in the continuous state approach. For example, we can decide to use as a monetary unit the amount of $1000 and so automatically, the underlying state space is discrete; moreover if we are sure that the maximum amount will be no more than one billion dollars, we also have a finite state space. Bühlmann (1994) also gives some advantages of working with discrete state space.

Furthermore, from an applied point of view, discrete time is in general enough to get results; for example, depending on our kind of investment, we are interested to know the results after an hour or a day or, maybe a year, and not necessarily instant after instant.
Of course, we know continuous time models can generally give more elegant mathematical developments as is shown in examples in Chapters 3 and 4, but we also know that the numerical results obtained in Chapter 4 needed discrete time and finite state space.
Also, as before, relation (1.25) represents the finite state space and furthermore:

$$t \in \mathbb{N}, \tag{1.30}$$

and the semi-Markov model used here still has (1.27) as embedded MC.
We also need the conditional probabilities

$$b_{ij}(t) = P[X_{n+1} = j, T_{n+1} - T_n = t \mid X_n = i], i, j \in I, \tag{1.31}$$

representing the probability that, after a time t from the nth transition, the value of the financial operation is S_j, given that the value of the operation is S_i at time T_n and that the ($n+1$)th transition happens after T_n in a time just equal to t.
Relations (1.19) of Chapter 4 give the evolution equation of the DTHSMP.

1.6 An Example Of Asset Evaluation

Now we seek to examine stochastic process problems that can explain the dynamic stochastic development of financial operations.
We would like to apply our model to a general stochastic financial operation, the purchase of goods or shares by an investor who would like to sell them for a profit.
First of all, we have to observe that we are in the hypothesis that the time is discrete if the quotations are fixed at the end of every stock exchange day, for example.
We also suppose that the investor is interested in a medium term investment, in the sense that he doesn't want to buy for a short period of speculation.
For this reason, the time unit will represent a month and "t" will give the number of months from a starting date. Furthermore, as we said previously, we suppose that the state space is finite, i.e.,

$$I = \{S_1, S_2, ..., S_m\}. \tag{1.32}$$

From the general results of Chapter 3, the model is characterized by the semi-Markov kernel **Q** equivalently by

$$p_{ij}, F_{ij}, i, j = 1, ..., m, \tag{1.33}$$

$F_{ij}(t)$ representing the increasing distribution function of waiting time τ_{ij}, in the sense that the asset value becomes S_j starting from S_i.

The probabilities $\phi_{ij}(t), i, j = 1,...,m$ of the associated semi-Markov process Z_t, giving here the probabilities that the asset value, once it gets the value S_i will have the value S_j after a time *t,* satisfy the system (1.29).

1.7 Two Transient Case Examples

Now we can study the dynamic evolution of the asset value we are examining as indeed, we know that
if the asset value is, at time 0, S_i, after *t* stock exchange days for example, the distribution of this asset on the possible values:

$$S_j,\ j = 1,\ldots,m, \tag{1.34}$$

is given by

$$\phi_{ij}(t),\ j = 1,...,m, t = 1,2,... \tag{1.35}$$

With this probability distribution of the asset value at time *t* we can very easily compute the expected value and variance of the asset at each time *t*.
Furthermore it is also very simple to compute the so-called VaR (Value of Risk) values of our financial operation, representing the value of the random variable with a probability less than a fixed threshold.
For example, the threshold can be fixed equal to 5%; then we know the r.v. value corresponding to this probability and so with a probability of 95% the loss value will be less than or equal to this value.
In this part we present two examples.
In the first example we apply our model to the evaluation of a real estate investment. The second will consider an application to the evaluation in time of the value evolution of a share.
Example 1 *A real estate investment example***:** we suppose that the price of the considered real estate can have the following values also representing the states of our model:

state	S_1	S_2	S_3	S_4	S_5	S_6
value	100000	120000	140000	160000	180000	200000

Table 1.1: real estate possible values

We suppose that we will follow the evolution of the system for 10 years with the year as time unit and let us assume that, based on the data, we solve the semi-Markov evolution equations (1.29).
Probabilities:

$$\phi_{ij}(t),\ i, j \in \{1,\ldots,6\}, t = 0,1,\ldots,10 \tag{1.36}$$

are reported in **Tables 1.2.1**, **1.2.2** and **1.2.3** for years 1, 5, 10:

	1	2	3	4	5	6
$\phi_{1j}(1)$	0.9561	0.0216	0.0083	0.0098	0.001	0.0032
$\phi_{2j}(1)$	0.0212	0.9228	0.0273	0.0111	0.0153	0.0023
$\phi_{3j}(1)$	0.0055	0.0352	0.9055	0.0378	0.0074	0.0086
$\phi_{4j}(1)$	0.004	0.0086	0.0253	0.9308	0.0211	0.0102
$\phi_{5j}(1)$	0.0004	0.0017	0.0157	0.0054	0.9636	0.0132
$\phi_{6j}(1)$	0.0011	0.0006	0.0229	0.0224	0.0197	0.9333

Table 1.2.1: probabilities to go from state *i* to state *j* at year 1

	1	2	3	4	5	6
$\phi_{1j}(5)$	0.7091	0.1483	0.0659	0.0367	0.0236	0.0164
$\phi_{2j}(5)$	0.0526	0.6908	0.0976	0.0625	0.0507	0.0458
$\phi_{3j}(5)$	0.0598	0.1196	0.5997	0.1131	0.0615	0.0463
$\phi_{4j}(5)$	0.0542	0.0717	0.1211	0.6512	0.0574	0.0444
$\phi_{5j}(5)$	0.0134	0.0474	0.0696	0.1153	0.6312	0.123
$\phi_{6j}(5)$	0.0169	0.0167	0.115	0.065	0.1228	0.6636

Table 1.2.2: probabilities to go from state *i* to state *j* at year 5

	1	2	3	4	5	6
$\phi_{1j}(10)$	0.2957	0.2294	0.2265	0.1099	0.0865	0.052
$\phi_{2j}(10)$	0.1626	0.2354	0.1875	0.1811	0.1313	0.1021
$\phi_{3j}(10)$	0.1291	0.2024	0.2311	0.196	0.1288	0.1126
$\phi_{4j}(10)$	0.1148	0.1584	0.1906	0.2939	0.1391	0.1032
$\phi_{5j}(10)$	0.045	0.1077	0.1279	0.2244	0.272	0.223
$\phi_{6j}(10)$	0.0624	0.0575	0.1681	0.1419	0.253	0.3171

Table 1.2.3: probabilities to go from state *i* to state *j* at year 10

Knowing the numerical values (1.36), it is possible to obtain the mean value $M(i,t)$ of the real estate at time $t=1,\ldots,10$ knowing the starting state at time 0 is I,

$$M(i,t)=\sum_{j=1}^{m}\phi_{ij}(t)S_j. \tag{1.37}$$

Table 1.3 gives these results.

	1	2	3	4	5	6
$M(i,0)$	100000	120000	140000	160000	180000	200000
$M(i,1)$	101754	121667	140641	159737	179393	197173
$M(i,2)$	104042	122679	141087	158269	177451	195632
$M(i,3)$	106820	125739	142371	156802	176681	191751
$M(i,4)$	108798	127910	142137	155225	174741	187728
$M(i,5)$	111334	130103	142714	154384	173452	185018
$M(i,6)$	115481	132933	143699	153874	171843	183358
$M(i,7)$	119359	135206	143944	152400	170439	179058
$M(i,8)$	123772	137549	145079	151410	167756	174217
$M(i,9)$	128202	140478	146244	150298	166249	171552
$M(i,10)$	132367	143784	146612	149870	164792	168340

Table 1.3: Real estate mean value

In **Table 1.4** the standard deviations related to each element of **Table 1.3** are given as risk measures of the investment.

	1	2	3	4	5	6
0	0	0	0	0	0	0
1	9633	10202	8968	7930	6378	11545
2	13139	12309	13396	12228	11866	14865
3	17049	17284	17574	15768	15157	19838
4	19444	20711	18905	18571	18194	23373
5	21990	23097	21128	20512	20501	24867
6	24820	26294	23206	22328	22733	25674
7	26417	27853	24650	24159	24699	27381
8	28044	29402	27282	26284	26043	28636
9	29234	30327	28977	28337	27212	29620
10	29604	31462	30554	29346	28775	30378

Table 1.4: Standard deviations of the real estate mean values

In an investment it is important to know the present value at time 0 of the mean result at time *t*. In this way it is possible to compare the investment with other financial operations; furthermore it gives a measure of the suitability of the investment. These results are given in **Table 1.5**.

	1	2	3	4	5	6
$Mpv(0)$	100000	120000	140000	160000	180000	200000
$Mpv(1)$	98791	118124	136545	155085	174168	191430
$Mpv(2)$	98069	115637	132988	149184	167264	184402
$Mpv(3)$	97755	115069	130290	143496	161688	175479
$Mpv(4)$	96666	113647	126287	137916	155255	166794

Mpv(5)	96038	112228	123106	133173	149621	159598
Mpv(6)	96713	111329	120346	128867	143916	153560
Mpv(7)	97050	109935	117039	123915	138583	145591
Mpv(8)	97707	108582	114527	119524	132428	137528
Mpv(9)	98256	107664	112084	115191	127416	131480
Mpv(10)	98493	106988	109093	111517	122621	125260

Table 1.5: Real estate mean present value

Furthermore, to have a clear idea of a real estate investment, the annuity rent given by the real estate should also be taken into account.

Example 2 *A share evolution example:* the second example is devoted to the evolution of a share.
We suppose that the share will have on our time horizon, the following possible values:

state	value	state	value	state	value
1	1000	8	1350	15	1700
2	1050	9	1400	16	1750
3	1100	10	1450	17	1800
4	1150	11	1500	18	1850
5	1200	12	1550	19	1900
6	1250	13	1600	20	1950
7	1300	14	1650	21	2000

Table 1.6: states of share evolution.

Here too, we only give results based on our data.
In this case the tables of results are bigger than in the previous examples.
In fact we will follow the evolution of the share value for two years with a time interval of one month. So we will only report the tables of the final results without presenting any intermediate result.
The mean values of the share for each period are reported in **Table 1.7**
As in **Table 1.3,** the rows correspond to the years and the columns to the starting state.
As risk measures of the financial operation, the variances of the mean values of the previous table are reported in **Table 1.8**.
Table1.9 gives the mean present values of the elements of **Table 1.7** and finally **Table 1.10** contains the VaR values at 95% of each of the elements of **Table 1.7**.

1000	1050	1100	1150	1200	1250	1300	1350	1400	1450	1500	1550	1600	1650	1700	1750	1800	1850	1900	1950	2000
1014.5	1062.89	1109.58	1161.36	1207.69	1256.57	1304.12	1355.55	1402.51	1450.97	1503.38	1548.16	1596.51	1646.19	1695.19	1742.85	1793.58	1840.3	1890.51	1937.34	1989.43
1023.83	1075.1	1120.29	1173.83	1213.14	1258.19	1308.63	1358.85	1406.74	1453.85	1503.81	1546.04	1592.14	1640.76	1689.78	1733.92	1789.35	1832.67	1881.5	1923.51	1975.82
1038.07	1086.44	1132.47	1183.86	1221.6	1264.03	1313.57	1364.86	1407.52	1455.17	1503.21	1545.39	1588.11	1634.19	1684.37	1727.55	1780.73	1820.74	1869.28	1909.59	1961.42
1049.43	1097.85	1142.78	1192.98	1229.6	1271.34	1319.75	1368.05	1410.16	1456.67	1504.04	1542.95	1586.94	1628.65	1680.51	1721.31	1773.17	1807.65	1857.5	1899.64	1947.44
1063.77	1108.61	1153.31	1203.23	1235.08	1275.83	1323.09	1373.07	1412.71	1457.81	1506.11	1541.25	1585.39	1625.04	1673.49	1714.55	1763.36	1795.66	1846.13	1885.16	1929.18
1078.93	1120.96	1164.66	1210.58	1241.52	1282.2	1327.53	1376.91	1414.05	1459.74	1507.79	1537.3	1581.89	1620.03	1666.29	1705.84	1755.49	1787.89	1836.12	1870.3	1917.68
1088.98	1134.24	1177.79	1220.56	1248.58	1289.75	1331.38	1383.05	1415.64	1461.98	1509.37	1535.34	1578.82	1616.37	1659.17	1697.41	1747.3	1776.62	1821.38	1857.26	1900.48
1103.64	1145.64	1189.82	1229.18	1254.85	1298.86	1337.09	1386.69	1417.72	1462.84	1506.9	1532.88	1574.58	1613.03	1653.45	1691.25	1737.02	1765.	1811.58	1839.83	1884.6
1119.84	1158.26	1201.32	1241.99	1263.42	1307.62	1343.22	1389.21	1421.5	1463.79	1509.27	1530.71	1571.86	1610.59	1647.51	1682.55	1728.72	1756.61	1799.89	1827.31	1867.89
1135.59	1169.66	1211.8	1253.79	1272.14	1312.76	1346.65	1396.41	1422.18	1466.12	1510.49	1528.12	1568.09	1607.08	1643.39	1672.88	1718.38	1747.92	1786.18	1813.46	1851.88
1149.24	1184.55	1222.2	1263.84	1279.42	1319.29	1350.9	1400.85	1425.22	1467.51	1509.76	1528.55	1563.26	1603.61	1637.4	1665.04	1708.68	1738.82	1774.13	1795.41	1837.99
1165.65	1195.16	1232.44	1273.58	1287.49	1327.86	1356.78	1404.56	1426.99	1470.37	1509.14	1526.93	1561.11	1600.71	1630.83	1658.04	1698.41	1725.64	1761.02	1782.42	1821.68
1181.17	1206.86	1245.52	1283.77	1297.03	1335.91	1363.17	1408.52	1430.24	1470.7	1508.21	1527.13	1556.74	1592.46	1624.19	1652.1	1690.18	1715.43	1749.04	1767.27	1806.3
1195.08	1222.34	1258.66	1292.37	1306.11	1344.58	1370.21	1414.64	1432.85	1472.93	1506.48	1526.54	1552.64	1585.87	1617.11	1645.93	1683.21	1703.14	1734.73	1753.95	1788.6
1210.28	1238.86	1274.09	1303.32	1314.19	1350.71	1372.38	1418.07	1434.77	1473.77	1502.78	1524.24	1548.46	1584.16	1611.15	1639.47	1671.38	1690.03	1724.54	1738.5	1770.63
1226.88	1254.28	1285.46	1314.45	1320.48	1359.32	1378.28	1422.16	1437.53	1474.32	1503.04	1523.94	1541.26	1578.55	1605.46	1628.53	1659.32	1678.07	1709.86	1722.28	1754.69
1242.91	1270.71	1299.89	1323.92	1330.13	1367.96	1385.08	1425.46	1438.43	1475.21	1502.43	1521.87	1538.46	1573.24	1600.87	1621.52	1651.01	1668.57	1695.13	1705.29	1736.34
1258.47	1284.69	1313.32	1333.98	1341.05	1374.14	1391.48	1430.22	1440.45	1477.01	1500.07	1519.02	1537.08	1565.71	1595.48	1613.66	1641.96	1654.83	1681.67	1688.62	1722.47
1274.81	1300.15	1326.9	1344.54	1348.23	1381.2	1396.43	1438.82	1442.78	1476.54	1498.2	1515.25	1532.16	1561.28	1588.36	1605.67	1633.79	1643.62	1667.23	1672.98	1708.04
1290.48	1314.64	1341.82	1356.01	1358.09	1388.31	1401.46	1443.28	1444.39	1477.73	1499.62	1514.2	1526.78	1558.14	1581.03	1595.82	1621.54	1629.88	1654.63	1657.31	1691.94
1309.31	1328.77	1355.2	1366.91	1367.73	1397.8	1408.43	1449.54	1445.28	1480.36	1498.08	1511.74	1523.1	1555.71	1574.69	1586.97	1608.85	1617.32	1639.9	1645.35	1675.3
1327.52	1343.89	1364.85	1379.34	1379.09	1407.74	1414.7	1455.23	1450.01	1480.46	1499.6	1509.06	1518.19	1551.12	1566.91	1578.13	1599.97	1602.98	1626.58	1627.23	1657.66
1347.51	1358.1	1379.42	1391.68	1389.8	1417.07	1422.64	1457.21	1455.98	1479.39	1502.23	1505.32	1512.78	1543.24	1556.09	1572.02	1590.78	1589.75	1611.48	1610.89	1636.66
1367.42	1373.41	1391.12	1406.36	1399.22	1425.24	1427.66	1462.89	1460.23	1479.44	1501.57	1501.39	1507.91	1537.1	1545.59	1561.65	1577.25	1575.77	1597.05	1595.47	1616.49

Table 1.7: share mean values

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6572.18	6515.78	4517.25	5070.16	3445.37	3582.59	3313.48	3068.11	3462.28	2966.2	2695.04	3066.44	3042.18	3856.84	3275.36	3696.48	4043.36	4577.35	5703.94	6919.05	5228.28
10731.8	12755.5	9671.56	11067.2	6624.02	5860.71	6792.18	6668.76	6726.48	7040.23	6251.35	6151.89	6651.73	8430.01	6413.24	8356.91	7013.77	8853.99	10493.7	13868.	11618.4
16226.5	17624.6	15418.4	15937.5	10853.6	9285.18	10213.6	10632.8	9280.65	9853.78	10174.1	9770.45	10451.	12804.7	10323.4	12134.9	12014.4	13837.7	16286.3	20971.6	17852.6
20781.3	23209.9	19731.1	20507.6	14910.2	12816.7	14301.5	13654.6	12079.1	13338.8	14036.4	12890.6	13230.4	16686.8	13488.6	16353.1	15674.9	19478.7	21863.3	25182.6	23687.3
26257.	26870.6	23835.2	25230.8	18037.8	15268.6	17111.5	17351.5	15417.7	16690.	17990.7	16055.	16469.	19672.1	17344.3	19528.4	19561.6	25064.	27151.6	31333.6	31044.7
32725.3	31503.2	28953.	28209.4	21295.1	18697.7	20683.	21095.3	19005.6	20965.5	21227.8	19849.2	19652.3	24088.9	21642.1	23774.1	22689.3	28452.1	32210.2	37599.5	35289.
36884.4	35597.3	34192.9	32678.	25228.1	23053.8	23480.6	25339.	22199.7	24487.9	23929.6	23362.7	22961.4	27747.	24710.7	27597.7	26385.4	33228.9	37869.	42115.4	41874.3
42293.	40372.7	39435.3	36225.7	28489.3	27270.2	26318.7	28371.9	24135.	27557.8	27895.6	26498.2	25930.4	31862.	27973.3	31000.8	30920.1	37395.3	42364.6	47805.7	47234.3
47238.7	44440.9	43397.9	40900.1	32381.3	31962.2	30064.2	31232.2	27378.9	30765.	32281.7	29682.1	27807.1	34431.9	31446.8	35175.7	34514.	41032.4	46384.5	52299.3	52082.4
51786.3	48159.3	47403.	45004.9	36180.1	34950.4	33177.8	34645.2	29781.7	34472.4	36230.9	32429.8	32329.1	37565.3	33714.	39041.3	39255.4	44485.4	50807.3	56121.1	56685.9
55580.4	52587.6	51134.6	48014.1	38822.5	38509.	36906.6	38195.5	32546.3	38077.9	39528.4	35715.8	35972.2	40696.1	36916.9	41901.9	43471.7	47821.3	55007.8	60971.2	60466.1
59405.7	55657.3	54740.3	50815.5	42025.7	42475.4	39802.7	40922.4	34615.5	41432.4	42942.2	39163.8	38076.1	43333.1	39971.9	45236.8	47572.7	51987.5	58509.1	64798.4	64159.9
63228.6	58335.3	58026.5	52972.2	45700.9	46598.2	42496.	43464.5	37169.7	43502.6	45605.7	41360.1	40966.6	48056.1	43113.4	48172.9	50585.3	55409.6	61742.6	68699.	67327.6
66401.5	61696.1	61364.7	55399.1	49619.8	50890.3	46034.8	46605.3	40562.8	46423.7	48614.3	44794.1	43449.3	51941.9	46813.2	50641.3	52750.9	58321.2	65926.9	70827.9	70645.
69114.	64669.8	64716.1	57961.	52686.8	53095.7	48954.5	49463.5	43042.7	49086.8	51028.9	47686.9	46673.3	55712.7	49968.1	53883.4	56472.7	62073.5	68188.3	73327.6	73443.8
71258.8	67529.1	66911.7	60241.7	55265.6	56335.7	52429.9	52731.8	46279.9	52389.5	53835.6	50851.2	50139.1	59499.	53187.9	57298.4	59826.8	64779.1	71052.2	76030.3	75311.5
73021.8	69614.	69617.8	62794.4	57947.5	59956.8	55396.4	55703.9	49091.1	54553.2	57523.3	54126.2	53576.4	62275.3	55948.2	59839.1	62346.7	67735.7	73337.3	77596.1	76559.3
74165.8	71056.8	71436.8	64932.3	60584.6	63033.6	59182.5	58401.4	52829.5	57545.	60252.	56799.1	56587.7	65436.9	59248.8	62568.1	65137.6	69822.2	76038.6	78817.2	78280.1
75614.8	72289.9	73066.1	66613.4	62565.1	65132.	62144.	61511.	56167.6	61413.	64529.3	60646.3	61067.3	67194.8	61779.	65004.7	67917.5	71738.8	77589.7	80287.6	79036.5
76574.8	72925.3	74053.2	68436.8	64247.5	68129.	63898.4	63899.7	58666.8	64854.4	67453.6	64060.5	64023.2	70847.1	64921.5	67878.5	71350.	73116.	78937.1	80509.	79182.
76623.1	73564.9	75074.6	70474.9	67618.4	70604.7	66962.5	66735.9	61540.9	68095.4	70144.	66607.5	66898.1	73963.4	67985.3	70539.8	73471.5	74620.2	80510.4	81180.5	79618.6
75598.	73791.7	75977.4	72495.7	69573.9	73520.	69823.1	69868.	65240.2	70617.1	73775.4	69503.8	69406.9	77528.6	70293.8	72096.7	75404.8	76165.9	81678.1	81017.3	79220.1
74453.6	74081.8	76791.1	73464.8	71801.7	76226.9	72656.5	72377.5	68108.8	73911.9	77131.3	73521.6	72702.7	79684.6	72467.7	73841.7	77644.6	76907.4	82296.6	80098.9	77647.9
72559.6	73260.4	76979.3	73758.1	74093.7	78125.8	76109.7	74640.5	70448.4	77597.2	79924.9	76393.7	75066.1	81909.9	74960.7	76857.6	78762.1	77157.9	82321.9	78842.7	75067.9

Table 1.8: variances of share mean values

1000	1050	1100	1150	1200	1250	1300	1350	1400	1450	1500	1550	1600	1650	1700	1750	1800	1850	1900	1950	2000
1011.26	1059.5	1106.04	1157.65	1203.84	1252.56	1299.96	1351.22	1398.04	1446.34	1498.58	1543.22	1591.42	1640.94	1689.78	1737.29	1787.86	1834.43	1884.48	1931.16	1983.08
1017.21	1068.14	1113.04	1166.24	1205.29	1250.05	1300.16	1350.06	1397.64	1444.44	1494.08	1536.04	1581.84	1630.15	1678.85	1722.71	1777.77	1820.82	1869.33	1911.07	1963.03
1027.92	1075.81	1121.39	1172.27	1209.65	1251.66	1300.72	1351.51	1393.75	1440.93	1488.5	1530.27	1572.57	1618.2	1667.89	1710.65	1763.31	1802.93	1850.99	1890.91	1942.22
1035.79	1083.58	1127.93	1177.47	1213.63	1254.82	1302.6	1350.27	1391.84	1437.74	1484.5	1522.9	1566.32	1607.49	1658.68	1698.94	1750.13	1784.16	1833.36	1874.96	1922.13
1046.6	1090.71	1134.7	1183.81	1215.14	1255.24	1301.73	1350.91	1389.91	1434.28	1481.8	1516.37	1559.8	1598.81	1646.47	1686.87	1734.9	1766.67	1816.33	1854.73	1898.04
1058.18	1099.41	1142.26	1187.3	1217.64	1257.54	1302.	1350.43	1386.86	1431.67	1478.79	1507.74	1551.47	1588.88	1634.25	1673.04	1721.73	1753.51	1800.81	1834.33	1880.8
1064.71	1108.97	1151.55	1193.36	1220.76	1261.01	1301.72	1352.24	1384.1	1429.41	1475.73	1501.13	1543.64	1580.36	1622.2	1659.59	1708.37	1737.04	1780.8	1815.88	1858.13
1075.64	1116.58	1159.63	1197.99	1223.01	1265.91	1303.16	1351.51	1381.75	1425.72	1468.66	1493.99	1534.63	1572.1	1611.5	1648.34	1692.95	1720.22	1765.62	1793.15	1836.79
1087.94	1125.26	1167.09	1206.61	1227.42	1270.36	1304.95	1349.63	1381.	1422.09	1466.27	1487.1	1527.08	1564.71	1600.57	1634.62	1679.47	1706.57	1748.61	1775.25	1814.67
1099.68	1132.68	1173.49	1214.14	1231.92	1271.25	1304.08	1352.26	1377.22	1419.77	1462.73	1479.81	1518.51	1556.27	1591.43	1619.98	1664.05	1692.65	1729.7	1756.12	1793.33
1109.29	1143.38	1179.72	1219.92	1234.95	1273.44	1303.95	1352.17	1375.68	1416.51	1457.29	1475.42	1508.93	1547.88	1580.5	1607.18	1649.29	1678.39	1712.47	1733.01	1774.11
1121.47	1149.87	1185.73	1225.31	1238.69	1277.54	1305.36	1351.32	1372.91	1414.65	1451.95	1469.06	1501.95	1540.05	1569.02	1595.2	1634.04	1660.24	1694.28	1714.86	1752.63
1132.78	1157.41	1194.49	1231.17	1243.89	1281.18	1307.32	1350.82	1371.64	1410.44	1446.42	1464.57	1492.96	1527.22	1557.65	1584.41	1620.93	1645.15	1677.38	1694.86	1732.3
1142.34	1168.4	1203.12	1235.35	1248.48	1285.25	1309.75	1352.22	1369.63	1407.93	1440.01	1459.19	1484.13	1515.89	1545.76	1573.31	1608.94	1627.99	1658.19	1676.56	1709.68
1153.02	1180.24	1213.81	1241.65	1252.	1286.8	1307.45	1350.97	1366.89	1404.04	1431.68	1452.12	1475.19	1509.21	1534.91	1561.89	1592.3	1610.07	1642.95	1656.24	1686.85
1165.04	1191.06	1220.68	1248.2	1253.93	1290.81	1308.82	1350.48	1365.08	1400.02	1427.28	1447.13	1463.58	1498.99	1524.54	1546.45	1575.69	1593.49	1623.68	1635.48	1666.25
1176.5	1202.82	1230.43	1253.18	1259.06	1294.87	1311.08	1349.3	1361.57	1396.39	1422.15	1440.56	1456.26	1489.18	1515.33	1534.88	1562.79	1579.41	1604.56	1614.18	1643.56
1187.49	1212.23	1239.25	1258.74	1265.41	1296.64	1312.99	1349.55	1359.21	1393.7	1415.46	1433.34	1450.39	1477.4	1505.49	1522.64	1549.35	1561.49	1586.82	1593.38	1625.32
1199.17	1223.	1248.17	1264.75	1268.23	1299.24	1313.57	1353.44	1357.17	1388.92	1409.3	1425.34	1441.24	1468.63	1494.11	1510.4	1536.84	1546.09	1568.3	1573.71	1606.69
1210.07	1232.73	1258.21	1271.52	1273.47	1301.81	1314.13	1353.35	1354.39	1385.65	1406.18	1419.85	1431.65	1461.05	1482.52	1496.39	1520.5	1528.32	1551.53	1554.04	1586.51
1223.8	1241.98	1266.69	1277.63	1278.4	1306.5	1316.44	1354.87	1350.89	1383.68	1400.24	1413.01	1423.62	1454.1	1471.85	1483.32	1503.77	1511.69	1532.8	1537.89	1565.88
1236.83	1252.08	1271.6	1285.1	1284.87	1311.56	1318.05	1355.81	1350.94	1379.31	1397.15	1405.96	1414.47	1445.15	1459.85	1470.31	1490.66	1493.46	1515.45	1516.06	1544.41
1251.38	1261.21	1281.02	1292.4	1290.65	1315.97	1321.15	1353.25	1352.12	1373.86	1395.06	1397.93	1404.86	1433.15	1445.08	1459.87	1477.3	1476.34	1496.52	1495.97	1519.9
1265.73	1271.28	1287.67	1301.78	1295.17	1319.25	1321.49	1354.1	1351.64	1369.42	1389.9	1389.73	1395.78	1422.79	1430.65	1445.52	1459.96	1458.59	1478.28	1476.82	1496.28

Table 1.9: share mean present values

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
947.783	47.5484	47.5802	47.6638	47.8153	47.9324	48.0689	47.9985	48.4068	48.2243	48.1627	48.4925	48.5067	48.6241	48.6904	48.8955	48.4637	49.1091	48.8454	49.1439	49.0315
946.	47.9545	47.7288	47.8514	48.5171	48.9013	48.7107	48.8915	49.1477	49.1634	49.005	49.6326	49.6867	49.914	58.108	119.578	49.4471	98.211	89.1273	232.992	193.978
943.12	48.2545	48.1192	48.381	49.0208	49.5477	49.5989	49.8675	64.5955	58.8508	81.4074	132.076	181.273	245.625	216.894	280.195	230.224	315.148	311.736	430.957	337.387
940.801	48.5764	48.2364	48.845	49.7083	58.2769	97.2378	89.6236	105.266	172.849	178.793	238.589	260.369	357.151	290.24	342.674	318.578	409.679	422.365	495.298	432.696
937.644	48.6423	48.2135	49.3846	81.4359	79.1177	152.953	135.245	173.984	238.056	236.546	292.405	303.632	402.153	365.903	413.759	378.433	487.471	494.981	558.203	552.264
934.56	49.1041	48.3095	49.8849	99.8053	94.0827	185.584	174.135	230.75	276.996	286.663	338.925	355.354	469.859	422.604	454.078	443.187	520.788	559.454	612.136	580.464
932.439	49.0449	48.563	63.2619	123.218	122.067	209.389	198.285	278.709	295.905	320.146	366.777	392.39	490.641	443.426	494.827	489.333	552.535	627.532	640.069	648.777
928.959	49.3624	49.3703	73.7247	133.63	142.58	214.862	225.37	296.942	315.277	369.373	394.902	416.862	507.703	476.922	524.207	532.589	579.553	659.821	666.451	683.343
924.563	49.6382	52.4832	77.7636	149.557	162.818	226.325	248.112	310.091	337.573	389.459	417.894	435.443	512.705	506.65	557.099	551.844	605.616	679.06	683.772	704.886
920.072	69.9029	60.8689	89.6694	157.396	186.031	234.295	257.407	319.417	356.06	407.207	430.944	463.07	523.477	516.229	585.492	586.855	632.528	695.782	694.358	725.064
915.246	53.3556	68.7843	97.3708	159.974	204.703	246.688	272.401	326.246	368.766	419.183	442.398	475.744	536.546	533.229	599.032	612.551	657.298	713.843	717.791	742.714
909.49	49.9068	71.0574	102.443	164.501	214.083	250.894	278.958	331.076	376.25	428.486	462.243	482.991	545.884	549.724	611.938	628.594	677.998	725.811	746.898	756.716
903.93	49.7852	77.3758	105.33	168.074	225.976	252.939	285.442	335.41	383.606	432.174	471.862	493.263	561.272	569.739	624.09	637.087	697.549	735.391	770.843	770.414
897.45	110.889	82.1214	115.248	174.007	236.79	259.274	292.499	342.959	390.498	439.083	487.7	504.09	569.877	587.114	632.649	644.244	708.838	747.75	783.448	787.528
888.704	218.253	84.2982	118.916	177.732	241.446	266.606	296.169	346.58	397.	446.073	502.817	513.775	576.013	596.68	638.808	660.074	720.662	755.18	799.041	797.445
877.538	205.235	88.1507	121.065	182.685	247.956	270.045	305.338	353.394	401.213	447.623	504.57	525.236	583.68	604.655	648.973	679.235	727.853	766.735	812.275	811.231
865.771	284.396	92.277	128.271	185.008	264.187	273.715	311.617	361.08	404.735	451.924	510.366	531.28	587.023	610.684	654.635	692.449	738.493	774.852	822.141	824.317
848.206	307.553	94.3188	129.707	184.842	340.014	278.463	317.641	367.246	406.585	456.743	514.393	536.787	592.985	619.29	662.4	703.754	747.529	785.698	831.616	837.408
832.516	371.553	96.9994	130.388	187.01	326.062	282.625	319.505	372.798	414.4	463.104	518.012	543.879	596.341	625.651	666.686	712.669	756.848	795.352	839.652	849.01
812.537	355.304	96.885	135.784	187.096	341.151	283.03	324.43	377.585	419.457	466.295	522.792	546.982	601.252	632.331	673.081	721.882	764.629	805.609	847.162	861.357
779.228	372.625	98.1206	141.453	189.814	326.494	284.688	325.962	382.496	421.889	468.133	524.123	553.153	605.932	639.267	678.163	727.714	770.626	816.866	853.776	872.809
724.286	420.253	99.2705	146.699	191.829	392.47	289.421	331.306	387.727	423.389	471.092	526.623	558.012	610.157	643.377	683.81	734.159	776.775	826.375	862.718	883.442
636.205	465.846	141.009	156.268	194.739	422.891	291.575	335.681	388.534	427.23	472.874	529.337	561.455	613.819	649.484	686.632	740.525	781.979	834.623	870.128	895.267
471.763	473.664	182.324	179.193	196.285	447.923	293.667	338.532	389.886	431.026	474.373	532.811	564.691	617.257	655.79	691.435	745.154	786.348	840.821	876.28	905.785

Table 1.10: VaR of the share mean values

1.8 Financial Application of Asymptotic Results

Assuming that the semi-Markov process is ergodic, from the results of Chapter 3, section 10, we know that:

$$\Pi_j = \frac{\pi_j \eta_j}{\sum_{h=1}^{m} \pi_h \eta_h}. \tag{1.38}$$

Let us recall that the probabilities $\pi_j, j = 1,...,m$ are the steady-state probabilities of the embedded Markov chain and η_j , j=1,...,m are the mean sojourn times in each state j, j=1,...,m.

We can give a double meaning to probabilities (1.38): on the one hand they represent the probabilities that the asset value is $S_j, j = 1,...,m$ after a long time period. On the other hand they give the percentages of time in which the asset will have values $S_j, j = 1,...,m$.

In a more general model we could be also interested in the probability that, at a given future time, the asset will assume a given value. This can be useful to resolve arbitrage problems and to decide when to buy or to sell the asset we are studying.

These entrance probabilities e_j in a given state $S_j, j = 1,...,m$ at a given time t, in the case in which our process has the steady state vector, have an ergodic behaviour and it is easy to compute them, in fact:

$$e_j = \frac{\Pi_j}{\eta_j} = \frac{\pi_j}{\sum_{h=1}^{m} \pi_h \eta_h}. \tag{1.39}$$

Now we can make some observations on the different meanings of (1.38) and (1.39).

We can say how it is possible to follow the dynamic evolution of an asset value by means of a homogeneous semi-Markov process directed by a transition probability matrix and by a sojourn time distribution function of the same asset value.

It must be made clear that we hypothetically assert that we don't know anything about the day quotation, but we know the embedded Markov chain matrix and the sojourn time distribution functions of our process.

In this case, we can suppose that the process works, without observation over a very long time period. Under these hypotheses we can formulate some questions:

- What probability should we assign to an event that the asset is likely to have on a given stock exchange day? The Π_j, j=1,...,m provide an answer..

- What probability should we assign to the event that the asset value becomes S_j on a given stock exchange day? To this question e_j, j=1,...,m is the answer.
- Supposing that the asset changes its value just that day; what probability should we give to the event that S_j becomes the asset value? In this case the answer is π_j (the element of the steady state vector).

Coming back to the last example we compute the three limit vectors.
The related results are reported in **Table 1.11**.

state	π	Π	e
1	0.038376	0.038786	0.003092
2	0.042548	0.042801	0.003429
3	0.047037	0.047356	0.003790
4	0.046367	0.046009	0.003736
5	0.048877	0.049879	0.003938
6	0.049470	0.049398	0.003986
7	0.054667	0.055429	0.004405
8	0.052218	0.051728	0.004208
9	0.055007	0.055255	0.004432
10	0.057753	0.057092	0.004654
11	0.056307	0.055002	0.004537
12	0.051747	0.051572	0.004170
13	0.051982	0.052283	0.004189
14	0.052682	0.051810	0.004245
15	0.045190	0.046125	0.003641
16	0.044767	0.045734	0.003607
17	0.045797	0.046545	0.003690
18	0.045992	0.045377	0.003706
19	0.040193	0.039611	0.003239
20	0.034717	0.034361	0.002797
21	0.038306	0.037844	0.003087

Table 1.11: limit vectors.

2 DISCRETE TIME MARKOV AND SEMI-MARKOV REWARD PROCESSES AND GENERALISED ANNUITIES

In Janssen and Manca (2006) the continuous time Markov reward processes (CTMRWP) were described. In Janssen and Manca (2004c) it is described how the DTMRWP can be seen as the natural stochastic generalization of the concept of discrete time annuity. In this section, after a short introduction of DTMRWP and their natural relation with annuities, we will show how it is possible to give a further generalisation to the annuity concept by means of SMRWP.

2.1 Annuities And Markov Reward Processes

The annuity concept is very simple and can be easier understood by means of **Figure 2.1** for the immediate case,

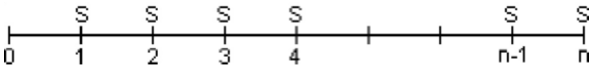

Figure 2.1: **constant rate annuity-immediate**.

and by **Figure 2.2** for the due case.

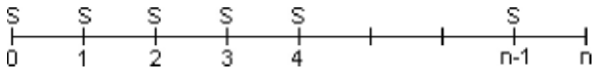

Figure 2.2: constant rate annuity-due.

Clearly the periodic payments can be variable. The simple problem to face is to compute the value at time 0 (*present value*) or at time *n* (*capitalisation value*) of the annuity. The present value formulas are presented at the beginning of Chapter 4. There exist similar results for capitalisation values (see Kellison (1991).
Instead of assuming as usual that S is deterministic, we will now consider S as r.v. with a set of possible values:

$$S = \{ S_1, S_2, \ldots, S_m \}. \tag{2.1}$$

Furthermore if we assume that the value at time k will depend only on the value at time $k-1$, we can model it with a Markov process and, as a sum or amount of money is associated with each state, we can use the framework given by the Markov reward model.
We will now give relations that describe the simplest case of discounted DTMRWP, trying to show in the immediate case the recursive nature of the process. Then the most two general relations in the immediate and due cases are given.

$$V_i^{(1)} = (1+r)^{-1}\psi_i = (1+r)^{-1}\psi_i, \tag{2.2}$$

$$V_i^{(2)} = (1+r)^{-1}\psi_i + \nu^2 \sum_{k=1}^{m} p_{ik}^{(1)}\psi_k = V_i^{(1)} + \nu^2 \sum_{k=1}^{m} p_{ik}^{(1)}\psi_k, \tag{2.3}$$

and in general:

$$V_i^{(n)} = V_i^{(n-1)} + \nu^n \sum_{k=1}^{m} p_{ik}^{(n-1)}\psi_k, \tag{2.4}$$

$$\begin{aligned} V_i^{(n)}(s) = & \ V_i^{(n-1)}(s) \\ & + \nu(s,s+n)\sum_{k=1}^{m} p_{ik}^{(n-1)}(s)\sum_{j=1}^{n} p_{kj}(s+n)\left(\gamma_{kj}(s,s+n)+\psi_{kj}(s,s+n)\right). \end{aligned} \qquad (2.5)$$

$$\ddot{V}_i^{(n)} = \ddot{V}_i^{(n-1)} + \nu(n-1)\sum_{k=1}^{m} p_{ik}^{(n-2)}\sum_{j=1}^{m} p_{kj}\psi_{kj}(n-1) + \nu(n)\sum_{k=1}^{m} p_{ik}^{(n-1)}\sum_{j=1}^{m} p_{kj}\gamma_{kj}(n). \qquad (2.6)$$

In the first case we have only fixed permanence rewards, in the other two cases we have variable permanence, transition rewards and interest rates.

Figure 2.3 can be associated to relation (2.4). It will be possible also to describe the due case by means of a similar figure.

The figure clearly shows that MRWP can be considered a natural generalisation of the annuity concept.

Any case of MRWP can be seen as a generalisation of the annuity concept.

We think that the connection between Markov reward processes and annuities is natural and that an annuity can be seen as the Markov reward process with only one state and only permanent rewards; for more details see Janssen and Manca (2004c).

In this light, in the finance environment, we can define *Markov reward processes* as *stochastic annuities.*

This first step also allows us to generalize also the payments of the annuities. In fact as we saw before, the rewards can be of permanence and of transition

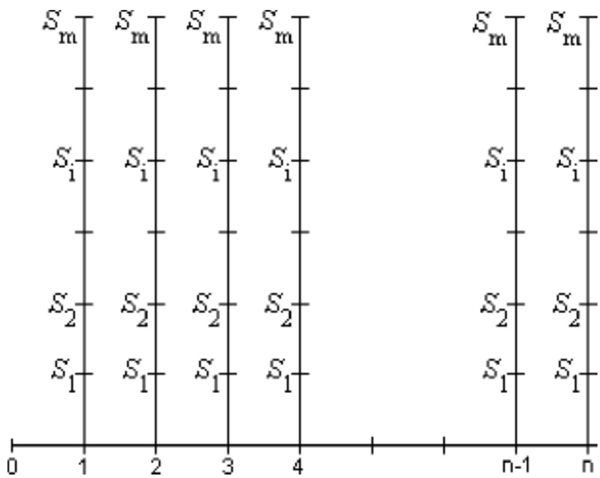

Figure 2.3: constant immediate stochastic annuity

(impulse reward) types; furthermore the permanence rewards can be independent from the transition or they can depend on the transition. All these kinds of rewards can be fixed or can vary with time.

In the case of simple annuity the rate can vary only because of time; in the case of *stochastic annuity*, clearly it can vary in the same way as the rewards because the rewards represent the generalization of the payments.

2.2 HSMRWP And Stochastic Annuities Generalization

This section will develop the semi-Markov extension of stochastic annuities.
For the sake of simplification the discrete time case will be discussed before the continuous one.
As an example of a potential application, it is known that, usually, an insurance contract makes provision with the premiums paid by the insured person to pay the claim amounts and to distribute some benefits to the shareholders or to some public organization. The benefit level but also the premium level could change as a function of the situation (state) in which the insured person may be.
In our opinion an insurance contract is a typical example of a generalized stochastic annuity (GSA) as defined below.

Definition 2.1 *A generalized stochastic annuity* (*GSA*) *is an annuity in which the payments are a function of the state of the system and the time of the transitions among the states is stochastic.*

The difference between a generalized stochastic annuity and a stochastic annuity, in discrete time environment, is in the fact that the time of the next transition is a random variable.
A GSA can be *homogeneous* or *non-homogeneous*.
It is *homogeneous* if the randomness of the time depends only on the duration between two transitions and it is *non-homogeneous* if the transitions are functions of the running time.
The simplest cases, as already shown, can be treated in a homogeneous environment; in some more composite cases it will require the non-homogeneous environment
The formulas of a GSA in the homogeneous case are the ones given for the continuous case in Chapter 4.
Janssen and Manca (2006) give models that could be used to construct a "GSA" for the insurance problem, using the simple examples given in Haberman and Pitacco (1999).
From all these examples we think that the most interesting is the one depicted in **Figure 2.4**. in which the weighted directed graph related to the first matrix is given.
To give an example of a GSA we will develop a case giving four different sets of data. In the four cases the following two different embedded Markov chains will be used:

$$\mathbf{P}=\begin{bmatrix}0.9 & 0.096 & 0.004\\ 0.4 & 0.592 & 0.008\\ 0 & 0 & 1\end{bmatrix}\quad \mathbf{P}'=\begin{bmatrix}0.9 & 0.096 & 0.004\\ 0.8 & 0.192 & 0.008\\ 0 & 0 & 1\end{bmatrix}. \tag{2.7}$$

For each one of these matrices, we will consider a case without any constraint on the waiting time distribution functions of the waiting (called *case* 1) and another case with as constraints (called *case* 2):

$$F_{ij}(1)\geq 0.5,\ i,j=1,\ldots,3. \tag{2.8}$$

The permanence reward vector will be:

$$\boldsymbol{\psi}=\begin{bmatrix}-200\\ 1200\\ 0\end{bmatrix}. \tag{2.9}$$

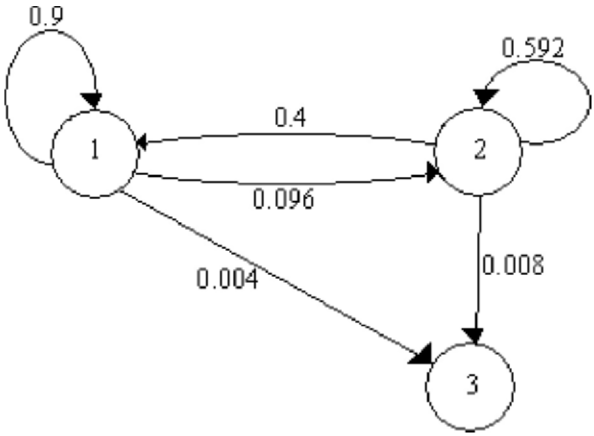

Figure 2.4

The value -200 is the yearly premium that the insured person will pay for his/her insurance and

1200 is the benefit that she/he will receive from the insurance company when she/he is disabled.

In the dead state, the permanence reward will be 0.

Using a DTHSMRW model, the following relations are obtained:

$$\begin{aligned}\ddot{V}_1(t) &= \left(1-H_1(t)\right)\ddot{a}_{\overline{t}|\delta}\psi_1+\sum_{k=1}^{3}\sum_{\vartheta=1}^{t}b_{1k}(\vartheta)\ddot{a}_{\overline{\vartheta}|\delta}\psi_1\\ &+\sum_{k=1}^{2}\sum_{\vartheta=1}^{t}b_{1k}(\vartheta)e^{-\delta\vartheta}V_k(t-\vartheta),\end{aligned} \tag{2.10}$$

$$\ddot{V}_2(t) = \left(1 - S_2(t)\right)\ddot{a}_{\overline{t|}\delta}\psi_2 + \sum_{k=1}^{3}\sum_{\vartheta=1}^{t} b_{2k}(\vartheta)\ddot{a}_{\overline{\vartheta|}\delta}\psi_2 + \sum_{k=1}^{2}\sum_{\vartheta=1}^{t} b_{2k}(\vartheta)e^{-\delta\vartheta}V_k(t-\vartheta), \tag{2.11}$$

$$\ddot{V}_3(t) = 0, \quad t = 1,\ldots,T. \tag{2.12}$$

Our time horizon is ten years and we only consider the due case, as usually premiums are paid at the beginning of each insured period.

The d.f. F_{ij}, i,j=1,2,3 will be constructed in the program by means of a pseudorandom number generator taking into account the given constraints; some of their values are given in **Table 2.1.1** and **Table 2.1.2** for the two considered cases:

Matrix **F**

F(1)			**F**(3)			**F**(5)		
0.1514	0.144	0.1398	0.3884	0.3767	0.3937	0.5917	0.4724	0.6387
0.0617	0.1675	0.0031	0.3556	0.189	0.1474	0.4568	0.3163	0.3041
1	1	0	1	1	0	1	1	0
F(7)			**F**(9)			**F**(10)		
0.744	0.634	0.8232	0.8838	0.8668	0.871	0.9232	0.9384	0.9581
0.6772	0.5729	0.5415	0.855	0.8002	0.8431	0.9527	0.9884	0.9774
1	1	0	1	1	0	1	1	0

Table 2.1.1:case 1

Matrix **F'**

F′(1)			**F**′(3)			**F**′(5)		
0.5763	0.5712	0.5675	0.7034	0.694	0.6987	0.8125	0.7445	0.8254
0.5273	0.5794	0.4969	0.6801	0.5902	0.57	0.7327	0.654	0.6494
1	1	0	1	1	0	1	1	0
F′(7)			**F**′(9)			**F**′(10)		
0.8941	0.8297	0.9207	0.9691	0.9526	0.9455	0.9902	0.9904	0.9905
0.8473	0.7827	0.7698	0.9397	0.8966	0.9226	0.9905	0.991	0.9907
1	1	0	1	1	0	1	1	0

Table 2.1.2: case 2

Table 2.2 presents the results in the four cases (s.s. means starting state).

I Case 1 and matrix **P**				*II Case 1 and matrix* **P′**			
time	s.s. 1	s.s. 2	s.s. 3		s.s. 1	s.s. 2	s.s. 3
1	-200	1200	0	1	-200	1200	0
2	-375	2331	0	2	-375	2298	0
3	-531	3330	0	3	-532	3168	0
4	-664	4234	0	4	-667	3897	0
5	-784	5070	0	5	-791	4532	0
6	-892	5864	0	6	-905	5122	0

7	-973	6584	0	7	-995	5598	0
8	-1045	7215	0	8	-1079	5947	0
9	-1103	7755	0	9	-1151	6167	0
10	-1136	8239	0	10	-1203	6332	0
III *Case 2 and matrix* **P**				IV *Case 2 and matrix* **P′**			
	s.s. 1	s.s. 2	s.s. 3		s.s. 1	s.s. 2	s.s. 3
1	-200	1200	0	1	-200	1200	0
2	-319	2074	0	2	-319	1787	0
3	-406	2789	0	3	-422	2224	0
4	-473	3419	0	4	-512	2586	0
5	-526	3994	0	5	-595	2900	0
6	-566	4535	0	6	-670	3188	0
7	-587	5023	0	7	-728	3416	0
8	-595	5447	0	8	780	3576	0
9	-591	5804	0	9	-826	3665	0
10	-572	6118	0	10	-860	3726	0

Table 2.2

The reader can easily see that the waiting time d.f. plays a very important role in the SMP environment and moreover that a consistent change of probabilities in the embedded Markov chain also gives differences in the results.
The considered constraint on the waiting time d.f gives a sensible decrease of the waiting time and explains a bigger change in the results.

3 SEMI-MARKOV MODEL FOR INTEREST RATE STRUCTURE

3.1 The Deterministic Environment

Usually, in a deterministic environment, relation (1.15) implies that:

$$\frac{S_2}{S_1} = \frac{\varphi(t_1,t_2,S_1)}{S_1} = \begin{cases} >1 & \text{if} \quad t_1 < t_2, \\ =1 & \text{if} \quad t_1 = t_2, \\ <1 & \text{if} \quad t_1 > t_2. \end{cases} \tag{3.1}$$

This ratio can be seen as a *transformation coefficient* giving the value at time t_2 of a unity account available at time t_1.
Starting now from (1.14) where S_2 depends on S_1, we have to suppose that

$$(S_1,t_1) \approx (S_2,t_2) \Leftrightarrow (kS_1,t_1) \approx (kS_2,t_2) \;\; \forall k \in \mathbb{R}^+ . \tag{3.2}$$

Then with $k = \dfrac{1}{S_1}$, it results that:

$$(S_1,t_1) \approx (S_2,t_2) \Leftrightarrow (1,t_1) \approx \left(\frac{S_2}{S_1},t_2 \right), \forall k \in \mathbb{R}^+ \quad (3.3)$$

or from the dynamic point of view:

$$S_2 = \varphi(t_1,t_2,S_1) \Leftrightarrow S_2 = S_1\varphi(t_1,t_2,1) . \quad (3.4)$$

This last relation means that it is possible to ignore the third variable and to simply write:

$$S_2 = S_1 z(t_1,t_2) . \quad (3.5)$$

Also taking into account time homogeneity, the transformation coefficient becomes:

$$S_2 = S_1 z(t_2 - t_1) = S_1 z(t) . \quad (3.6)$$

If $t > 0$, then $z(t) > 1$ and it results that:

$$r(t) = z(t) - 1 , \quad (3.7)$$

and so in this special case, the interest rate $r(t)$ at time t is defined as in the classical way.

3.2 The Homogeneous Stochastic Interest Rate Approach

Even with the preceding simplifications, starting with the homogeneity assumption with respect to time and sums, the interest rate remains one of the most unpredictable objects.

But as in the case of sums, it is possible to assume that the rate will vary inside a finite interval:

$$r \in [\, r', r'' \,] . \quad (3.8)$$

Under the condition (3.8), we start with a model having continuous state space, but for the sake of applications, as in the previous section, we will work with only a finite number of states, i.e.:

$$r \in I = \{ r_1, r_2, \ldots, r_m \} . \quad (3.9)$$

From theoretical and numerical points of view, this assumption implies strong simplification.

For real life applications, the interest rates are usually fixed in a discrete range, with as the smallest unit the basic point having for value 0.01%.

As specified above, time will also be on a discrete scale and by means of DTHSMP of kernel **Q,** we will be able to follow the time evolution of the interest rate in a given time horizon.

The evolution of the interest rate in time gives a yield curve and term structure of implied forward rates.

This time the functions $F_{ij}(t)$ represent waiting time d.f. of the interest rate change from state r_i to state r_j, $i,j=1,\ldots,m$.

The function $\phi_{ij}(t), i, j \in I$ of our semi-Markov process in discrete time process $Z = (Z_t, t = 0,1,...,T)$ represents the probability that at time *t* the interest rate will be r_j, given that the interest rate was r_i at time 0.
As before, this probability is obtained by the sum of the two terms:

$$\delta_{ij}(1 - H_i(t)) \quad i, j = 1,2,...,m, \tag{3.10}$$

giving the probability of remaining in the starting initial interest rate without any change in [0, *t*] and has meaning only if $i = j$ and the second term,

$$\sum_{h=1}^{m} \sum_{\theta=0}^{t} b_{ih}(\theta)\phi_{hj}(t-\theta), \tag{3.11}$$

giving the probability that the interest rate is equal to r_j after a time *t*, given that it was r_i at time 0 and it reaches the new value with at least one state change (the first one in θ).
If $\Gamma_i(\theta)$ represents the r.v. interest rate related to the period $(\theta - 1, \ \theta) \subset (0, \ T)$ with values $r_j \in I$ and $\phi_{ij}(\theta)$ its conditional probability distribution, given that at time 0 the interest rate was r_i, we get the mean interest rate and, as risk measure, the related variance:

$$E(\Gamma_i(\theta)) = \sum_{j=1}^{m} \phi_{ij}(\theta) r_j \tag{3.12}$$

and:

$$\sigma^2(\Gamma_i(\theta)) = \sum_{j=1}^{m} \phi_{ij}(\theta) r_j^2 - \left(\sum_{j=1}^{m} \phi_{ij}(\theta) r_j \right)^2. \tag{3.13}$$

3.3 Discount factors

The quantity

$$\Upsilon_i(\theta) = \left(1 + \Gamma_i(\theta)\right)^{-1} \tag{3.14}$$

is the random discount factor related to the period $[\theta - 1, \ \theta] \subseteq [0, \ h]$, depending on the period and

$$A_i(h) = \prod_{\theta=1}^{h} \Upsilon_i(\theta). \tag{3.15}$$

The independence hypothesis among the random discount factors $\Upsilon_i(\theta)$ is assumed.
This assumption of the independence of stochastic interest rates r_i can be considered equivalent to the independence of the increments of invested sums.
Now the expected values and the variances are given by:

$$E(\Upsilon_i(\theta)) = \sum_{j=1}^{m} \phi_{ij}(\theta)(1+r_j)^{-1}, \tag{3.16}$$

$$\nu_i(h) = E(A_i(h)) = \prod_{\theta=1}^{h} E(\Upsilon_i(\theta)), \tag{3.17}$$

and for the second ones, generalising the computation of $\sigma^2(XY)$:

$$\sigma^2(\Upsilon_i(\theta)) = \sum_{j=1}^{m} \phi_{ij}(\theta)(1+r_j)^{-2} - \left(\sum_{j=1}^{m} \phi_{ij}(\theta)(1+r_j)^{-1} \right)^2, \tag{3.18}$$

$$\sigma^2(A_i(h)) = \sum_{\theta=1}^{h} \sum_{q=1}^{\binom{h}{\theta}} S_{C_q} M_{D_q}, \tag{3.19}$$

where:

$$S_{C_q} = \prod_{r=1}^{\theta} \sigma_i^2(\zeta_r), \qquad (\zeta_1,\ldots,\zeta_\theta) = C_q,$$
$$M_{D_q} = \begin{cases} \prod_{r=1}^{h-\theta} \mu_i^2(\eta_r), & (\eta_1,\ldots,\eta_{h-\theta}) = D_q \quad \text{if} \quad \theta < h, \\ 1 & \text{if} \quad \theta = h, \end{cases} \tag{3.20}$$

$C_q \in \mathbf{C}(h,\theta)$, where $\mathbf{C}(h,\theta)$ is the set of all θ–combinations of the set $\{1,\ldots,h\}$, and

$$\begin{cases} \sigma_i^2(\lambda) = \sigma^2(\Upsilon_i(\lambda)), \\ \mu_i(\lambda) = E(\Upsilon_i(\lambda)), \\ C_q \cup D_q = \{1,\ldots,h\}. \end{cases} \tag{3.21}$$

Once the $\gamma_i(h)$ are known, it is possible to evaluate the mean present value of a given financial operation that begins at time 0 and ends at time h.
Clearly, if the value at time 0 is known, the mean value at time h will be obtained multiplying the initial value by $1/\nu_i(h)$.

The knowledge of the expected value and variance of $A_i(h)$ allows important applications in *risk control*. In particular it allows decisions and choices in financial projects by using the mean-variance criterion.
Instead of directly using relation (3.19) for computation, we can use the formula for the computation of the two independent variables iteratively as follows: computing first the variance of the first two variables, then considering successively the result as a variance of one variable and doing the computation of the variance of two variables, taking the one obtained from the first two and the third variable and so on.

3.4 An Applied Example In The Homogeneous Case

To illustrate numerically the preceding model, let us suppose that we are interested in the dynamic evolution of a stochastic interest rate whose possible values are restricted to the ones given in **Table 3.1** and on a ten year time horizon The first row of the table contains the states and the second row the related interest rate values.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
.03	.035	.04	.045	.05	.055	.06	.065	.07	.075	.08	.085	.09	.095	.1

Table 3.1: homogeneous discount factor states

In this way we have 15 states and 11 time periods, from time period 0 up to the time period 10.
To get results we constructed a "Mathematica" program able to solve DTHSMP based on the following data:

a) the transition matrix $\mathbf{P}$, embedded Markov chain in DTHSMP;

b) the matrix $\mathbf{F}(t)$, waiting time distribution functions.

As explained in Chapter 4 the matrix values can be obtained by means of observation on real data; in this example we filled up both matrices with pseudorandom numbers.
The transition matrix $\mathbf{P}$ is a square matrix of order 15.
Filling up this matrix we supposed that the transition probabilities were bigger in the three mean diagonals and that they decrease moving away from them.
The matrix $\mathbf{F}(t)$ is formed by 11 square matrices, each one of order 15. It results that:

$$F_{ij}(t) > 0 \quad , \quad i, j \in \{1,\dots,15\}, \quad t \in \{1,\dots,10\}. \tag{3.22}$$

As we are working in the transient case, with a 10 year horizon time, we consider all these distribution functions trimmed at the last period.
After the construction of the embedded Markov chain and the distribution functions we were able to apply the DTHSMP, which is solved as described in Chapter 4 to get the following probability distributions:

$$\phi_{i,1}(t),\dots,\phi_{i,15}(t) \text{ with } i \in \{1,\dots,15\} \quad \text{and } t \in \{0,\dots,10\} \tag{3.23}$$

to compute for each *i, t* the interest rate mean values and related standard deviations.
The results of the example tables are reported on the following pages.
The rows of each table represent the time and the columns the states of the system at initial time.
The uni-period mean discount factors $E\left(V_i(\theta)\right)$ are reported in **Table 3.2**, the mean discount factors form 0 to h, $E(A_i(h))$ in **Table 3.3** and the variances $\sigma^2(A_i(h))$ in **Table 3.4**.

0.970874	0.966184	0.961538	0.956938	0.952381	0.947867	0.943396	0.938967	0.934579	0.930233	0.925926	0.921659	0.917431	0.913242	0.909091
0.967834	0.963952	0.959871	0.954966	0.951058	0.947124	0.943565	0.938807	0.934519	0.930374	0.926371	0.922583	0.91947	0.915053	0.912265
0.966126	0.961405	0.957064	0.952999	0.950031	0.946533	0.943076	0.939106	0.934207	0.930947	0.927442	0.924542	0.9214	0.917353	0.91547
0.963326	0.958571	0.955605	0.950878	0.948539	0.946098	0.943022	0.939675	0.934709	0.931608	0.929053	0.926567	0.924382	0.919572	0.918374
0.960459	0.956538	0.953222	0.949263	0.947079	0.94484	0.942835	0.940019	0.934983	0.932656	0.930632	0.928074	0.92601	0.921708	0.921443
0.956534	0.953551	0.950704	0.946779	0.946144	0.944408	0.942355	0.940199	0.935582	0.933285	0.931397	0.92931	0.928188	0.924314	0.923706
0.953602	0.95085	0.949108	0.944971	0.944508	0.944098	0.941906	0.940553	0.936193	0.934423	0.933139	0.931208	0.930043	0.926644	0.926447
0.950446	0.947508	0.946316	0.943464	0.943165	0.942983	0.941164	0.94089	0.936765	0.935547	0.934297	0.932776	0.9323	0.929164	0.930848
0.947292	0.945308	0.944165	0.941842	0.942081	0.941992	0.940713	0.941686	0.936948	0.936432	0.935065	0.933705	0.934052	0.931154	0.933762
0.944473	0.942833	0.94222	0.94034	0.940513	0.940335	0.940581	0.942167	0.937448	0.937052	0.936094	0.935236	0.935166	0.933812	0.935385
0.941415	0.940885	0.940494	0.939893	0.939669	0.939558	0.940124	0.941743	0.938139	0.937278	0.937101	0.937208	0.937041	0.936491	0.937187

Table 3.2: uniperiod mean discount factors

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0.970874	0.966184	0.961538	0.956938	0.952381	0.947867	0.943396	0.938967	0.934579	0.930233	0.925926	0.921659	0.917431	0.913242	0.909091
0.939645	0.931355	0.922953	0.913843	0.90577	0.897748	0.890156	0.881509	0.873382	0.865464	0.857751	0.850307	0.84355	0.835665	0.829332
0.907815	0.895409	0.883325	0.870892	0.860509	0.849748	0.839485	0.827831	0.81592	0.805702	0.795515	0.786144	0.777247	0.7666	0.759228
0.874522	0.858314	0.844109	0.828112	0.816226	0.803945	0.791652	0.777892	0.762648	0.750599	0.739075	0.728415	0.718473	0.704943	0.697256
0.839942	0.821009	0.804624	0.786096	0.77303	0.759599	0.746397	0.731233	0.713063	0.70005	0.687807	0.676023	0.665313	0.649752	0.642481
0.803434	0.782874	0.764959	0.74426	0.731398	0.717372	0.703371	0.687505	0.667129	0.653346	0.640622	0.628235	0.617536	0.600575	0.593464
0.766156	0.744396	0.726028	0.703304	0.690811	0.677269	0.66251	0.646634	0.624561	0.610501	0.597789	0.585017	0.574335	0.556519	0.549813
0.72819	0.705321	0.687052	0.663542	0.651549	0.638654	0.623531	0.608412	0.585067	0.571153	0.558513	0.54569	0.535453	0.517097	0.511792
0.689808	0.666745	0.64869	0.624951	0.613812	0.601606	0.586564	0.572933	0.548177	0.534846	0.522246	0.509514	0.500141	0.481497	0.477892
0.651506	0.628629	0.611209	0.587667	0.577298	0.565712	0.551711	0.539799	0.513888	0.501178	0.488871	0.476516	0.467715	0.449628	0.447013

Table 3.3: mean discount factors

0	0	0	0	0	0	0	0	0	0	0	0
0.000121441	0.0000875335	0.0000634564	0.0000754516	0.0000406411	0.0000345933	0.0000183074	0.0000357202	0.0000386126	0.0000239687	0.0000208211	0.000034074
0.000287545	0.000254133	0.000208187	0.000195748	0.000113064	0.0000899575	0.0000607827	0.0000855019	0.0000825102	0.0000662219	0.0000758755	0.00011846
0.000502217	0.000475784	0.000367085	0.000335449	0.000210234	0.000163186	0.000121062	0.000159251	0.000137462	0.00013079	0.000152881	0.000220658
0.000732416	0.000698349	0.000544865	0.000487609	0.00030888	0.000255168	0.000183803	0.00023488	0.000201156	0.000204365	0.000238098	0.000318877
0.000971872	0.000925021	0.000720374	0.000637577	0.000399517	0.000344436	0.000243511	0.000306819	0.000275606	0.000280195	0.000314284	0.000401214
0.00118222	0.00113128	0.000873032	0.000765185	0.000483609	0.000424423	0.000309455	0.00036958	0.000342802	0.000352327	0.000386423	0.000477559
0.00135279	0.00130412	0.00100076	0.000871497	0.000561982	0.000497477	0.000378313	0.000439124	0.000406648	0.000417649	0.000448476	0.000541008
0.00147351	0.00142183	0.00109645	0.00094908	0.000629146	0.000565882	0.000445714	0.000501717	0.000465813	0.00047359	0.00049863	0.000585076
0.00153587	0.00148559	0.00115265	0.000999735	0.000683603	0.00062468	0.000506678	0.00055886	0.000513853	0.00052353	0.000538535	0.000619049
0.00154178	0.00150845	0.00118028	0.00102993	0.000723073	0.000673813	0.000562152	0.000604382	0.000553528	0.000561326	0.000569663	0.000641578

Table 3.4: variance of discount factors

3.5 A Factor Discount Example In The Non-Homogeneous Case

As the interest rate and the discount factors are fundamentally non-homogeneous phenomena, it is interesting to see that the same kind of example can be provided also in a non-homogeneous environments.
The state values and their numbers were changed mainly because the number of results in the non-homogeneous case is by far larger and in this way the results can be given easily.
All the remarks given in sections 3.2 and 3.3 hold in the non-homogeneous environment and to repeat all the formulas for this case could be tedious.
This time we have nine states given in **Table 3.5**.

States	1	2	3	4	5	6	7	8	9
int rate	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
disc fact	0.9901	0.9852	0.9804	0.9756	0.9709	0.9662	0.9615	0.9569	0.9524

Table 3.5: discount factor model state

The non-homogeneous uni-period mean discount factors obtained with the expected value using the transition probabilities $\phi_{ij}(s,t)$, a solution of the DTNHSMP, are given in **Table 3.6.**
The elements in the first column give the couple (s,t) and the corresponding mean uni-periodic discount factor.

Time	1	2	3	4	5	6	7	8	9
0-1	0.9901	0.9852	0.9804	0.9756	0.9709	0.9662	0.9615	0.9569	0.9524
0-2	0.9887	0.9845	0.9795	0.9755	0.9713	0.9665	0.9619	0.9575	0.9543
0-3	0.9879	0.9828	0.9783	0.9754	0.9712	0.9668	0.9626	0.9584	0.9556
0-4	0.9868	0.9817	0.9773	0.9751	0.9714	0.9673	0.9630	0.9593	0.9570
0-5	0.9856	0.9805	0.9768	0.9751	0.9716	0.9674	0.9637	0.9602	0.9581
0-6	0.9836	0.9791	0.9762	0.9744	0.9715	0.9679	0.9642	0.9615	0.9595
0-7	0.9823	0.9779	0.9756	0.9740	0.9711	0.9685	0.9647	0.9625	0.9614
0-8	0.9806	0.9767	0.9747	0.9735	0.9710	0.9685	0.9658	0.9633	0.9626
0-9	0.9793	0.9759	0.9742	0.9731	0.9709	0.9688	0.9663	0.9647	0.9644
0-10	0.9780	0.9749	0.9737	0.9727	0.9713	0.9693	0.9671	0.9661	0.9662
0-11	0.9760	0.9737	0.9729	0.9720	0.9714	0.9697	0.9687	0.9676	0.9673
3-4	0.9901	0.9852	0.9804	0.9756	0.9709	0.9662	0.9615	0.9569	0.9524
3-5	0.9883	0.9844	0.9798	0.9751	0.9712	0.9669	0.9620	0.9582	0.9552
3-6	0.9866	0.9833	0.9786	0.9745	0.9717	0.9677	0.9624	0.9600	0.9569
3-7	0.9844	0.9819	0.9776	0.9746	0.9716	0.9679	0.9636	0.9615	0.9588
3-8	0.9830	0.9800	0.9764	0.9738	0.9715	0.9682	0.9655	0.9627	0.9600
3-9	0.9809	0.9789	0.9760	0.9731	0.9714	0.9689	0.9662	0.9640	0.9617
3-10	0.9783	0.9775	0.9746	0.9727	0.9711	0.9701	0.9677	0.9650	0.9634
3-11	0.9758	0.9748	0.9739	0.9719	0.9715	0.9704	0.9693	0.9670	0.9659
6-7	0.9901	0.9852	0.9804	0.9756	0.9709	0.9662	0.9615	0.9569	0.9524
6-8	0.9880	0.9833	0.9787	0.9744	0.9705	0.9668	0.9628	0.9592	0.9558
6-9	0.9857	0.9807	0.9766	0.9731	0.9708	0.9673	0.9653	0.9622	0.9588
6-10	0.9822	0.9787	0.9754	0.9734	0.9714	0.9681	0.9662	0.9649	0.9621
6-11	0.9777	0.9754	0.9745	0.9729	0.9715	0.9698	0.9673	0.9669	0.9648

Table 3.6: non-homogeneous uni-period discount factors

Table 3.7 gives the mean discount factors related to all the considered time periods.

Time	1	2	3	4	5	6	7	8	9
0-1	0.9901	0.9852	0.9804	0.9756	0.9709	0.9662	0.9615	0.9569	0.9524
0-2	0.9789	0.9700	0.9603	0.9517	0.9430	0.9338	0.9249	0.9163	0.9089
0-3	0.9670	0.9532	0.9395	0.9283	0.9159	0.9029	0.8904	0.8781	0.8685
0-4	0.9543	0.9358	0.9182	0.9052	0.8897	0.8733	0.8575	0.8424	0.8312
0-5	0.9406	0.9175	0.8969	0.8827	0.8645	0.8449	0.8264	0.8089	0.7964
0-6	0.9251	0.8984	0.8755	0.8600	0.8399	0.8177	0.7967	0.7777	0.7642
0-7	0.9088	0.8785	0.8541	0.8377	0.8156	0.7919	0.7686	0.7485	0.7346
0-8	0.8911	0.8580	0.8325	0.8155	0.7919	0.7670	0.7424	0.7211	0.7072
0-9	0.8726	0.8374	0.8110	0.7935	0.7688	0.7430	0.7174	0.6956	0.6820
0-10	0.8535	0.8163	0.7897	0.7718	0.7468	0.7202	0.6938	0.6720	0.6589
3-4	0.9901	0.9852	0.9804	0.9756	0.9709	0.9662	0.9615	0.9569	0.9524
3-5	0.9785	0.9699	0.9606	0.9513	0.9429	0.9343	0.9250	0.9169	0.9097
3-6	0.9653	0.9537	0.9401	0.9271	0.9163	0.9041	0.8902	0.8803	0.8705
3-7	0.9503	0.9364	0.9190	0.9035	0.8902	0.8751	0.8578	0.8464	0.8346
3-8	0.9341	0.9177	0.8974	0.8798	0.8649	0.8472	0.8282	0.8148	0.8012
3-9	0.9163	0.8983	0.8759	0.8562	0.8401	0.8209	0.8002	0.7854	0.7705
3-10	0.8964	0.8781	0.8536	0.8328	0.8159	0.7963	0.7744	0.7580	0.7423
6-7	0.9901	0.9852	0.9804	0.9756	0.9709	0.9662	0.9615	0.9569	0.9524
6-8	0.9782	0.9687	0.9595	0.9506	0.9423	0.9341	0.9258	0.9179	0.9103
6-9	0.9641	0.9501	0.9371	0.9250	0.9148	0.9036	0.8936	0.8832	0.8728
6-10	0.9470	0.9298	0.9140	0.9005	0.8886	0.8748	0.8634	0.8522	0.8397

Table 3.7: non-homogeneous discount factors

The last results reported in**Table 3.8** give the variance matrix related to the discount factors of **Table 3.7**

time	1	2	3	4	5	6	7	8	9
0-1	0.00003	0.00002	0.00002	0.00001	0.00001	0.00001	0.00001	0.00001	0.00004
0-2	0.00007	0.00007	0.00006	0.00003	0.00002	0.00003	0.00003	0.00003	0.00011
0-3	0.00012	0.00015	0.00011	0.00006	0.00005	0.00006	0.00006	0.00007	0.00017
0-4	0.00019	0.00023	0.00016	0.00009	0.00008	0.00010	0.00009	0.00011	0.00024
0-5	0.00028	0.00032	0.00022	0.00013	0.00011	0.00015	0.00012	0.00015	0.00031
0-6	0.00037	0.00042	0.00029	0.00018	0.00015	0.00020	0.00016	0.00020	0.00038
0-7	0.00047	0.00051	0.00035	0.00024	0.00020	0.00024	0.00020	0.00025	0.00044
0-8	0.00056	0.00059	0.00042	0.00029	0.00025	0.00029	0.00024	0.00030	0.00049
0-9	0.00065	0.00067	0.00048	0.00034	0.00030	0.00034	0.00029	0.00035	0.00054
0-10	0.00073	0.00073	0.00054	0.00040	0.00035	0.00039	0.00034	0.00039	0.00058
3-4	0.00004	0.00002	0.00001	0.00002	0.00002	0.00003	0.00001	0.00003	0.00006
3-5	0.00011	0.00006	0.00005	0.00006	0.00004	0.00007	0.00003	0.00007	0.00015
3-6	0.00021	0.00012	0.00011	0.00011	0.00008	0.00012	0.00007	0.00013	0.00024
3-7	0.00032	0.00021	0.00017	0.00017	0.00013	0.00017	0.00013	0.00020	0.00032
3-8	0.00043	0.00029	0.00024	0.00023	0.00019	0.00022	0.00019	0.00026	0.00040
3-9	0.00055	0.00038	0.00031	0.00030	0.00025	0.00028	0.00026	0.00032	0.00047
3-10	0.00066	0.00047	0.00039	0.00036	0.00031	0.00035	0.00032	0.00038	0.00053
6-7	0.00005	0.00005	0.00004	0.00003	0.00002	0.00003	0.00003	0.00004	0.00008
6-8	0.00012	0.00012	0.00013	0.00008	0.00006	0.00007	0.00010	0.00012	0.00018
6-9	0.00022	0.00022	0.00022	0.00014	0.00012	0.00014	0.00018	0.00022	0.00028
6-10	0.00035	0.00032	0.00032	0.00023	0.00020	0.00022	0.00026	0.00031	0.00038

Table 3.8: variance of non-homogeneous discount factors

4 FUTURE PRICING MODEL

The discrete time Homogeneous Semi-Markov Process can also be used for pricing futures.
Let us consider an asset observed on a discrete time scale

$$\{0,1,\ldots,t,\ldots,T\}, \qquad T<\infty \tag{4.1}$$

having $Y(t)$ as t market value of the future contract at time t.
In order to make the basic stochastic process

$$\{Y(t); \quad t=0,1,\ldots,T\} \tag{4.2}$$

suitable to this model, it is supposed that the future has minimal and maximal values so that the set of all possible values is the closed interval $[Y_{\min}, Y_{\max}]$ partitioned in a set of m states. By letting

$$\Delta=\frac{Y_{\max}-Y_{\min}}{m-1}, \tag{4.3}$$

the state space is given by:

$$\begin{aligned} I &= \{Y_{\min},\ Y_{\min}+\Delta,\ Y_{\min}+2\Delta,\ldots,Y_{\min}+(m-2)\Delta,\ Y_{\max}\},\\ &= \{S_1,\ S_2,\ldots,S_m\}. \end{aligned} \tag{4.4}$$

The next step is the description of how it is possible to follow the time evolution of a future contract by means of a DTHSMP.
Application of the homogeneous semi-Markov model requires that the state transitions follow a homogeneous finite Markov process. Moreover it is supposed that the waiting time in the state before a transition follows a discrete Markov process. The last hypothesis introduces randomness in the waiting times in the sense that, when the process arrives at a state, it can stay in this state for a random time. The evolution equation represents the probability that the process, once arrived at state i, representing the future value S_i, will be in state j at time t. This peculiarity distinguishes the semi-Markov approach from all other approaches in literature.
Indeed, most of the pricing models usually only introduce the Markov assumption for state transitions.
As the states $\{S_1,S_2,\ldots,S_m\}$ represent possible future values, it is not necessary to use the interest rates, and no assumption of the term structure of interest rates is needed as, for example in the cost of carry model (CCM), one of the most widely used models in the literature of pricing futures contracts, using randomness by giving a stochastic term structure of interest rates.
The model considered, as already mentioned, is homogeneous in time.

The process evolution equation can be explained by reference to the application shown here. Given the set of states *I*, that represents the set of possible prices of the future contract, by supposing that, at time 0, the future price is S_i, the process evolution equation gives the probability of the future price being S_j at time *t*. Such a probability is given by the two addends: $d_{ij}(t)$ (this is different from 0 if and only if *i=j*) represents the probability that the future price is equal to the starting price without any change in the state within a time *t*. The second addend represents the probability that the future price is S_j at time *t* and that it arrives in this state having changed states at least once.

The $\phi_{i*}(t)$ represents a distribution function. Then it is also possible to compute the expectation $E[Y(t)]$, the variance and the value at risk (VaR).

Note that the randomness, by means of a semi-Markovian approach, is introduced over the length of the investment. This fact, as far as the authors know, has never been considered in derivative pricing literature.

4.1 Description Of Data

A total 7,408 records which refer to the primary future stock index market (Fib30) bought from March 27th 1998 to September 17th 1998 (expiry date) have been analysed. Each record was filled by data contained in the following fields: Date, Operation (buy or sell), Contract amount, Price, Customer identification.

All the prices of the contracts, expressed in Italian lire, belong to the range [27,955; 39,490]. In order to reconstruct precisely all the market movements of future traders, the file uses only units in the field of contract amount. In such a way, 10,394 unit financial operations made by 36 different traders are obtained.

T	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9
#	5917	1495	707	549	445	331	209	153	86

Table 4.1.1: duration days of contract I part

T	9-10	10-11	11-12	12-13	13-14	15-16	17-17
#	6	6	6	6	10	6	6

Table 4.1.2: duration days of contract II part

The holding period of the future contract varies between a minimum of one day, i.e., the trader holds the asset for a period variable from 0 to 24 hours (intraday), and a maximum of 17 days. As shown in **Table 4.1**, 5.917 contracts belong to the former subclass, and only six contracts belong to the latter.

The holding periods are summarized in **Table 4.1**, where the # symbol represents the number of the futures contracts and *t* the holding periods, expressed in days.

In order to simplify the problem, the time is discretized into nine subclasses of one day each, and contracts held for a time longer than eight days are also added to the last subclass. Moreover, the future prices are discretized into 122 subclasses that are the states of the stochastic process. Each subclass is equal to 100 Italian lire and in the first class, we find the futures priced at less than 28,000 Italian lire; in the last, all the contracts having prices ranging from 40,001 to 40,100 Italian lire. We used Italian lire because we get real data for this example.

4.2 The Input Model

In order to implement the solving procedure, we need the following inputs:
m=122 (number of states of SMP).
T=9 (number of periods examined for the transient analysis of SMP).
The transition matrix **P** of the embedded MC in SMP and the square lower-triangular block matrix ${}^{T}\mathbf{F}$ of order 10, the blocks of which are of order 122, are built as follows.
First, the starting file is divided into 38 different files referring to the traders. Then, the files are ordered on the date field and the first record with label B (buy) is supposed to match the first record with label S (sell). In other words, the first future sold by a certain trader, is the one bought from the same individual at the more recent date, and so on for the subsequent records.
In this way, a new file formed by the following fields for each trader is obtained: Buying date; Buying price; Sale date; Sale price.
Next, all the records related to a fixed holding period are taken from these files. In this way, nine files of the holding periods $\{0,1,\ldots,8\}$ are constructed, each file containing all the movements traders made during that fixed holding period, and ending with the buying price and the related sale price.
Then the matrices of the holding periods are filled with the frequencies taken from the previous files. In this way, nine square matrices of order 122, one for each holding period, of the frequencies of the future contracts bought and sold at the respective prices are obtained.
The elements of the matrices, the a_{ij}, are the frequencies of the futures bought at price *i* and sold at price *j*.
The transition matrix **P** of the embedded MC in SMP and the probability distribution functions stored in the square lower-triangular block matrix ${}^{T}\mathbf{F}$ of order 10, with blocks of order 122, are obtained as described in Chapter 4, section 4.
Note that the distribution functions used as inputs for matrix ${}^{T}\mathbf{F}$ are derived directly from the raw data. Usually in other SMP applications the standard distributions such as Poisson or Exponential were used, and only the parameters of such distributions were estimated by means of the raw data.

4.3 The Results

After solving the evolution equation of the semi-Markov model, a large amount of information can be obtained. For each time $t \in \{1,2,\ldots,T\}$ and for each starting state $i \in I = \{1,2,\ldots,m\}$, $\phi_{ij}(t)$ represents the probability distributions defined by the evolution equation DTHSMP. Unfortunately, it is not possible to show the obtained probability distributions because of the huge amount of data (133,956). They are, however, available upon request. After computing these probabilities distributions it is possible to compute some statistic indices useful to the investor.

The price at time 0 (starting state), the price expectation of the future contract at the expiry date (time 9), the corresponding sigma square, the present value at time zero considering a risk rate per year of 4% and the corresponding value at risk with 5% of probability are reported in **Table 4.2**.

By means of the expectation value, an investor may forecast his own return with a certain risk estimated by means of the standard deviation.

For the sake of accuracy, present values are also computed, although the time is very short. In order to do this, the financial operations traded in a unit period time are supposed to be made at midday. For example, the data related to the 9-th day is discounted for 8.5 unit periods.

The value at risk (VaR), already introduced in this chapter gives the investor the possibility of estimating a risky investment.

With the knowledge of the probability distributions $\phi_{ij}(t)$, it is possible to compute the value of the random variable with a probability less than a fixed threshold.

When a threshold is fixed equal to 5%, the values of the last column in **Table 4.2** provide the information that, with probability 0.95, the value of the future will be greater than the value obtained in column two of the same table, that is the VaR value at 95%.

As our r.v. are finite, first of all, we must compute the values of k such that:

$$\sum_{j=1}^{k} \phi_{ij}(t) < 0.05.$$

and

$$\sum_{j=1}^{k+1} \phi_{ij}(t) \geq 0.05.$$

Then, the values of the random variable corresponding to the kth and the k+1th state are linearly interpolated, obtaining the hypothetical value corresponding to a cumulated probability equal to 5%.

For brevity we report in **Table 4.2** only the first 15 and the last 15 rows of the results.

Finally, **Table 4.3** and **Table 4.4** present the same data that refers to time 5 and to time 1 (intraday trading) respectively.
As we know, the latter is particularly significant to the problem we have dealt with. Also in the last two tables, we report the first 15 and the last 15 rows of the results. All the data including the matrix intermediate results are available upon request.
The fact that the solution of the SMP evolution equation gives probability distributions allows the reader to easily obtain the dynamic evolution of the financial phenomenon of interest and to estimate the investment risk in different ways. Finally, the authors would like to draw attention to the simplicity of the model and of its use. It was, however, a rather complex procedure to turn the raw data into input matrices, since the financial data were ready to be used for the homogeneous Markovian models.

starting state	price at time 9	sigma square	p.v. at time 0	VaR
28,000	28,000	0	27,974	28,000
28,100	29,905	880	29,878	28,146
28,200	28,200	0	28,174	28,105
28,300	29,822	877	29,795	28,245
28,400	29,974	906	29,947	28,363
28,500	30,014	899	29,987	28,412
28,600	30,002	929	29,975	28,386
28,700	30,056	879	30,028	28,526
28,800	29,894	874	29,867	28,492
28,900	30,124	914	30,096	28,462
29,000	29,948	805	29,921	28,510
29,100	30,272	954	30,244	28,449
29,200	30,243	1,104	30,215	28,442
29,300	30,115	975	30,087	28,393
29,400	30,090	931	30,062	28,317
38,700	37,965	517	37,930	37,312
38,800	38,460	442	38,425	37,784
38,900	38,900	0	38,864	38,805
39,000	39,000	0	38,964	38,905
39,100	38,079	555	38,045	37,317
39,200	38,756	587	38,720	37,807
39,300	39,300	0	39,264	39,205
39,400	38,115	567	38,080	36,930
39,500	39,500	0	39,464	39,405
39,600	39,600	0	39,564	39,505
39,700	39,700	0	39,664	39,605
39,800	39,800	0	39,764	39,705
39,900	39,900	0	39,864	39,805
40,000	40,000	0	39,963	39,905
40,100	40,100	0	40,063	40,005

Table 4.2: Value at expiry date

starting state	price at time 5	sigma square	p.v. at time 0	VaR
28,000	28,000	0	27,986	28,000
28,100	29,024	626	29,010	28,343
28,200	28,200	0	28,186	28,105
28,300	29,193	445	29,179	28,536
28,400	29,317	727	29,303	28,401
28,500	29,452	411	29,438	28,706
28,600	29,624	631	29,610	28,571
28,700	29,863	548	29,848	29,001
28,800	29,686	412	29,671	29,303
28,900	29,706	521	29,692	28,826
29,000	29,964	401	29,949	29,426
29,100	29,978	566	29,963	29,048
29,200	29,769	1,071	29,754	28,099
29,300	29,962	821	29,947	28,791
29,400	29,897	807	29,882	28,770
38,700	38,615	288	38,597	37,930
38,800	38,813	286	38,794	38,215
38,900	38,900	0	38,881	38,805
39,000	39,000	0	38,981	38,905
39,100	38,539	307	38,520	37,850
39,200	38,818	471	38,799	38,025
39,300	39,300	0	39,281	39,205
39,400	38,178	325	38,159	37,829
39,500	39,500	0	39,481	39,405
39,600	39,600	0	39,581	39,505
39,700	39,700	0	39,681	39,605
39,800	39,800	0	39,781	39,705
39,900	39,900	0	39,881	39,805
40,000	40,000	0	39,981	39,905
40,100	40,100	0	40,081	40,005

Table 4.3: Value after five days

starting state	intraday price	sigma square	p.v. at time 0	VaR
28,000	28,000	0	27,998	28,000
28,100	28,484	112	28,482	28,303
28,200	28,200	0	28,198	28,105
28,300	28,732	122	28,730	28,504
28,400	28,563	222	28,562	28,315

28,500	28,644	150	28,642	28,410
28,600	28,926	340	28,925	28,514
28,700	28,892	200	28,890	28,610
28,800	28,800	0	28,798	28,705
28,900	29,017	293	29,015	28,806
29,000	29,000	0	28,998	28,905
29,100	29,356	504	29,354	29,007
29,200	29,608	527	29,606	29,110
29,300	29,684	288	29,682	29,214
29,400	29,784	450	29,782	29,310
38,700	38,719	39	38,717	38,606
38,800	38,877	158	38,875	38,706
38,900	38,900	0	38,898	38,805
39,000	39,000	0	38,998	38,905
39,100	39,100	0	39,098	39,005
39,200	39,227	45	39,225	39,107
39,300	39,300	0	39,298	39,205
39,400	39,400	0	39,398	39,305
39,500	39,500	0	39,498	39,405
39,600	39,600	0	39,598	39,505
39,700	39,700	0	39,698	39,605
39,800	39,800	0	39,798	39,705
39,900	39,900	0	39,898	39,805
40,000	40,000	0	39,998	39,905
40,100	40,100	0	40,098	40,005

Table 4.4: Intraday values

5 A SOCIAL SECURITY APPLICATION WITH REAL DATA

5.1 The Transient Case Study

The example is similar to the one given in Chapter 2, section 9.7 and in Chapter 3 section 13 but real life data will now be used with the same invalidity degrees.
Let us recall that the aim is to calculate in particular the average degree of disablement to be expected in given epochs, in view of determining the premiums to be paid to the insuring agency by employers in connection with disabling professional diseases.
We assume that the temporal evolution of the disabling disease is the same as in the examples mentioned above.
The data selected for the numerical experiment reflects the situation with disabling professional diseases in Campania (Italian region) from 1945 to 1978.

They were obtained from about 800 case histories of workers suffering from professional diseases.
The average degree of disablement is calculated according to the following expression:

$$\overline{S}_i(t) = \sum_{j=1}^{m} \phi_{ij}(t) S_j \tag{5.1}$$

for a fixed time *t*, with S_j representing the upper bound disability degree inside the *j*th state.
Result (5.1) is similar to (13.2) of Chapter 3, but here, we will obtain results in transient instead of immediately working asymptotically as before (see result 13.3 of Chapter 3).
Furthermore, for **P** and **F** we used the following estimators:

$$\tilde{p}_{ij} = \frac{n_{ij}}{n_i} \tag{5.2}$$

where n_{ij} is the number of transitions from S_i to S_j and n_i is the number of observed elements in S_i;

$$\tilde{F}_{ij}(t) = 1 - e^{-t/\tilde{\lambda}_{ij}} \tag{5.3}$$

where $\tilde{\lambda}_{ij}$ is the estimated mean sojourn time in state S_i given the state S_j successively visited.
The mean degree of disablement was calculated for epochs of 10, 20 and 30 years.
Table 5.1 shows matrix **P** and **Table 5.2** gives the degree of disablement at 10, 20 and 30 years, computed by means of (5.1).
The $\mathbf{\Phi}(t)$ matrices, here omitted for the sake of brevity, were calculated by the program at discrete epochs 1, 2, ..., 10, ..., 20, ...,30.
Table 5.3 shows the results obtained for the mean degree of disablement at 10, 20 and 30 years, that are practically identical to those obtained by (De Dominicis and Manca (1984b)) using the same data with an asymptotic expression for $\phi_{ij}(t)$ as $t \to \infty$ and leading to an asymptotic mean degree of disablement of 82.75%.
The result is also consistent with that obtained by (De Dominicis and Manca (1984b)) using, in the asymptotic case, a stationary MC, yielding a mean degree of 82.85%.
On the basis of Dutch data, and using a stationary MC model, (Yntema (1965)) found a value of 79 %, as shown in Chapter 2, result (9.122))
In spite of the fact that the approaches followed in the two cases (transient and asymptotic) are quite different, it is interesting to note the fast convergence of the present method to the asymptotic value.

.00000	1.00000	.00000	.00000	.00000
.00000	.82629	.16012	.00906	.00453
.00273	.01639	.85247	.11475	.01366
.00000	.01869	.05607	.68225	.24299
.00000	.00000	.03279	.01639	.95082

Table 5.1: transition probabilities of the embedded Markov Chain

	years		
state	10	20	30
1	33.70	34.38	34.43
2	34.12	34.40	34.44
3	52.94	53.22	53.25
4	74.99	75.20	75.21
5	98.09	98.06	98.06

Table 5.2. Mean degree of disablement at 10, 20 and 30 years conditioned by the state of entry in the system.

Years	10	20	30
Mean degree	82.96	83.04	83.05

Table 5.3: **Total mean disability degree at 10, 20 and 30 years**

5.2 The Asymptotic Case

For the same data, we also study the asymptotic behaviour of the HSMP chain given before.

Figure 5.1 presents the graph related to the transition matrix **P** given in **Table 5.1.** From the graph it is easy to understand that the related Markov Chain is irreducible because it is possible to go from one state to all the other states.

In this case the stationary probability vector of the embedded Markov chain can be computed and is reported in **Table 5.4**.

1	2	3	4	5
0.00061	0.03669	0.22228	0.11365	0.62677

Table 5.4: Markov stationary probability vector

From the irreducibility of the Markov chain, it is possible to compute the $\phi_{ij}(\infty)$ that are given in **Table 5.5**.

1	2	3	4	5
0.000366	0.036774	0.222061	0.117726	0.623073

Table 5.5: semi-Markov stationary probability vector

Table 5.6 reports the unconditional mean sojourn times, with the year as time unit.

1	2	3	4	5
2.008219	3.353425	3.342466	3.465753	3.326027

Table 5.6:unconditional mean sojourn times

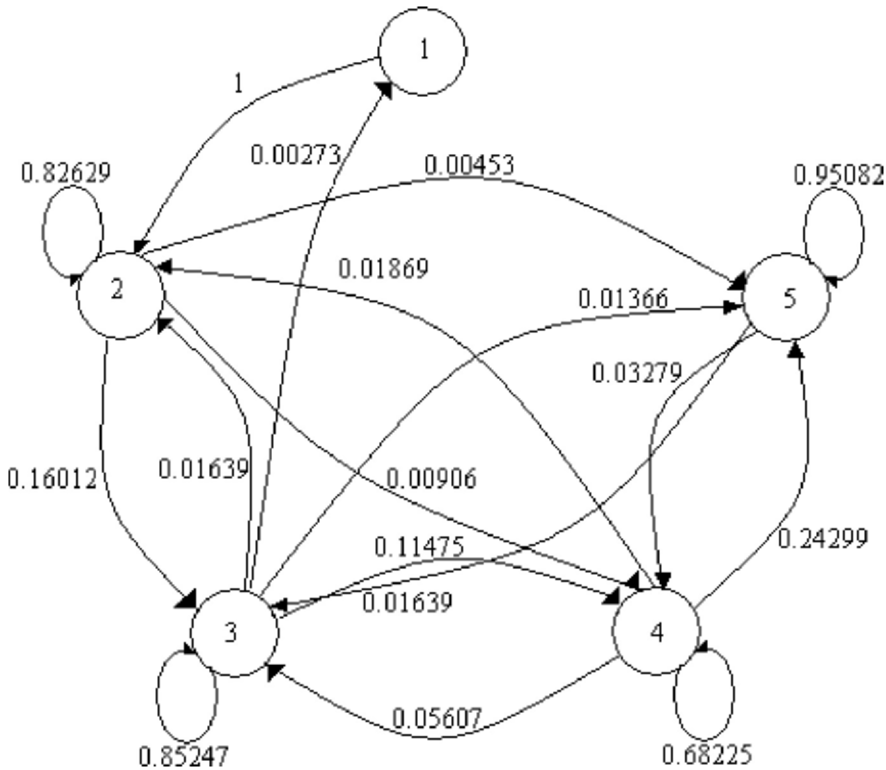

Figure 5.1: P-weighted transition graph

The last table of this section reports the asymptotic mean degrees obtained respectively using Markov and semi-Markov models, the results already given in the previous subsection.

Markov	Semi-Markov
82.8533	82.75804

Table 5.7: asymptotic mean degrees

6 SEMI-MARKOV REWARD MULTIPLE-LIFE INSURANCE MODELS

In this part, we present three different models for insurance applications.
The first two examples concern the topic *of multiple life problems* and are strictly connected with pension scheme problems.
These two examples are given to show how to write the related formulas and to attempt a first easy approach to pension scheme problems; for these reasons we do not tackle in this case the problem of input data.
The third example is a real-life case concerning the evolution of a *disability illness*. It is similar to the example developed in the previous section but the data are different and the example will be developed without using, as previously, a negative exponential increasing d.f. but with the distribution functions directly obtained from the observed data, and furthermore, we use the reward model.
The results will be the RMPV and the rewards in this case will be only of permanence type.
The first example will describe a *two-life annuity example*.
The typical case is the one of a retired person who can leave his/her pension to the spouse.
Though we want to introduce the age dependence of the pensioners in addition to the duration of pension, we will begin the simpler case of fixed death probabilities to develop the topic thoroughly in the non-homogeneous case.
In the non-homogeneous environment it is possible to take into account many aspects of pension schemes. Furthermore, as we will show in the last chapter, the extension of the non-homogeneous case gives the possibility to consider all the relevant aspects of pension schemes.
First we describe the model by means of a graph. This graphical approach was described in Manca (1988).
Figure 6.1 reports the multiple state graph related to our example. The states of the system are the following:
rs – state in which both the insured, retired and spouse, are living (*state* 1)
r – state in which only the direct pensioner is living (*state* 2)
s – state in which only the spouse is living (*state* 3)
d – state in which both the insured are dead (*state* 4).
The transition probabilities of the embedded Markov chain are:
p_{rs} - probability of surviving of both the insured
p_r - probability of surviving of direct pensioner
p_s - probability of surviving of the spouse
q_{rs} - probability of dying in the same period of both the insured persons
q_r - probability of dying of direct pensioner
q_s - probability of dying of the spouse.

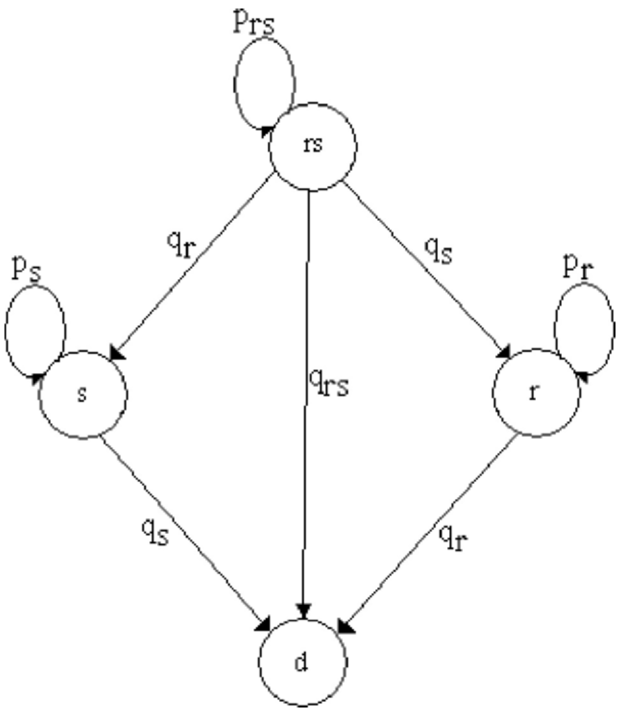

Figure 6.1: two-life annuity

The embedded Markov chain has the following form:

$$\mathbf{P}=\begin{bmatrix} p_{rs} & q_s & q_r & q_{rs} \\ 0 & p_r & 0 & q_r \\ 0 & 0 & p_s & q_s \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.1}$$

In the non-homogeneous case, the MC matrix structure is similar and the only difference is that the non-zero elements will be time dependent.
The $F_{ij}(t)$ and $F_{ij}(s,t)$ could be constructed by the available data.

In this case the rewards are only of the permanence type and transition rewards make no sense.
We suppose that permanence rewards will be constant in time and are paid at the end of the period, and also that the intensity of interest rate is fixed.
Under all these assumptions, it is possible to write the reward evolution equations in both the homogeneous and non-homogeneous cases as follows:

$$V_1(t) = \big(1 - H_1(t)\big)\bar{a}_{\overline{t}|\delta}\psi_1 + \sum_{k=1}^{4}\int_0^t \dot{Q}_{1k}(\vartheta)\bar{a}_{\overline{\vartheta}|\delta}\psi_1 d\vartheta$$
$$+ \sum_{k=1}^{3}\int_0^t \dot{Q}_{1k}(\vartheta)e^{-\delta\vartheta}V_k(t-\vartheta)d\vartheta, \tag{6.2}$$

$$V_1(s,t) = \big(1 - H_1(t)\big)\bar{a}_{\overline{t-s}|\delta}\psi_1 + \sum_{k=1}^{4}\int_s^t \dot{Q}_{1k}(s,\vartheta)\bar{a}_{\overline{\vartheta-s}|\delta}\psi_1 d\vartheta$$
$$+ \sum_{k=1}^{3}\int_s^t \dot{Q}_{1k}(s,\vartheta)e^{-\delta(\vartheta-s)}V_k(\vartheta,t)d\vartheta, \tag{6.3}$$

$$V_2(t) = \big(1 - H_2(t)\big)\bar{a}_{\overline{t}|\delta}\psi_2 + \sum_{k=2}^{4}\int_0^t \dot{Q}_{2k}(\vartheta)\bar{a}_{\overline{\vartheta}|\delta}\psi_2 d\vartheta$$
$$+ \int_0^t \dot{Q}_{22}(\vartheta)e^{-\delta\vartheta}V_2(t-\vartheta)d\vartheta, \tag{6.4}$$

$$V_2(s,t) = \big(1 - H_2(s,t)\big)\bar{a}_{\overline{t-s}|\delta}\psi_2 + \sum_{k=2}^{4}\int_s^t \dot{Q}_{2k}(s,\vartheta)\bar{a}_{\overline{\vartheta-s}|\delta}\psi_2 d\vartheta$$
$$+ \int_s^t \dot{Q}_{22}(s,\vartheta)e^{-\delta(\vartheta-s)}V_2(\vartheta,t)d\vartheta, \tag{6.5}$$

$$V_3(t) = \big(1 - H_3(t)\big)\bar{a}_{\overline{t}|\delta}\psi_3 + \sum_{k=3}^{4}\int_0^t \dot{Q}_{3k}(\vartheta)\bar{a}_{\overline{\vartheta}|\delta}\psi_3 d\vartheta$$
$$+ \int_0^t \dot{Q}_{33}(\vartheta)e^{-\delta\vartheta}V_3(t-\vartheta)d\vartheta, \tag{6.6}$$

$$V_3(s,t) = \big(1 - H_3(s,t)\big)\bar{a}_{\overline{t-s}|\delta}\psi_3 + \sum_{k=3}^{4}\int_s^t \dot{Q}_{3k}(s,\vartheta)\bar{a}_{\overline{\vartheta-s}|\delta}\psi_3 d\vartheta$$
$$+ \int_s^t \dot{Q}_{33}(s,\vartheta)e^{-\delta(\vartheta-s)}V_3(\vartheta,t)d\vartheta. \tag{6.7}$$

Relations(6.2) and (6.3) give the mean present value of all the rewards received by the system starting at time 0 or at time s from state 1 (both the insured are living at the given time). The sum of the last elements rises to three because the absorbing state doesn't give any benefit.

Relations (6.4), (6.5) and (6.6), (6.7) are similar to the previous ones. They give the mean present value of the rewards received starting at time 0 or s respectively from state 2 or 3. From both these states the only real transition is with state 4 (dead state).

The formulas take into account this property and the last element takes into account the only transition that is possible, the virtual one.

The next example will take into account a *three-life model*.

In this case, the example could be a pension to a family composed of three members, the direct pensioner, her/his spouse and their child. The child will be in the position to get the pension up to the end of his life (disabled child).

Figure 6.2 reports the multi-state graph related to this problem with as states:

rsc – state in which all the insured are living (*state 1*)
rs – state in which the pensioner and the spouse are living (*state 2*)
sc – state in which the spouse and the child are living (*state 3*)
rc – state in which pensioner and the child are living (*state 4*)
r – state in which only the direct pensioner is living (*state 5*)
s – state in which only the spouse is living (*state 6*)
c – state in which the child is living (*state 7*)
d – state in which all the insured are dead (state 8).
The transition probabilities of the embedded Markov chain are:
p_{rsc} - probability of surviving of all the insured people
p_{rs} - probability of surviving of the pensioner and the spouse
p_{sc} - probability of surviving of the spouse and the child
p_{rc} - probability of surviving of the pensioner and the child
p_r - probability of surviving of direct pensioner
p_s - probability of surviving of the spouse
p_c - probability of surviving of the child
q_{rsc} - probability of dying of all the insured people
q_{rs} - probability of dying in the same period of both the insured persons
q_{sc} - probability of dying of the spouse and the child
q_{rc} - probability of dying of the pensioner and the child
q_r - probability of dying of direct pensioner
q_s - probability of dying of the spouse.
q_c - probability of dying of the child

The homogeneous transition matrix of the embedded Markov chain is the following:

$$\mathbf{P} = \begin{bmatrix} p_{rsc} & q_c & q_r & q_s & q_{sc} & q_{rc} & q_{rs} & q_{rsc} \\ 0 & p_{rs} & 0 & 0 & q_s & q_r & 0 & q_{rs} \\ 0 & 0 & p_{sc} & 0 & 0 & q_c & q_s & q_{sc} \\ 0 & 0 & 0 & p_{rc} & q_c & 0 & q_r & q_{rc} \\ 0 & 0 & 0 & 0 & p_r & 0 & 0 & q_r \\ 0 & 0 & 0 & 0 & 0 & p_s & 0 & q_s \\ 0 & 0 & 0 & 0 & 0 & 0 & p_c & q_c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.8}$$

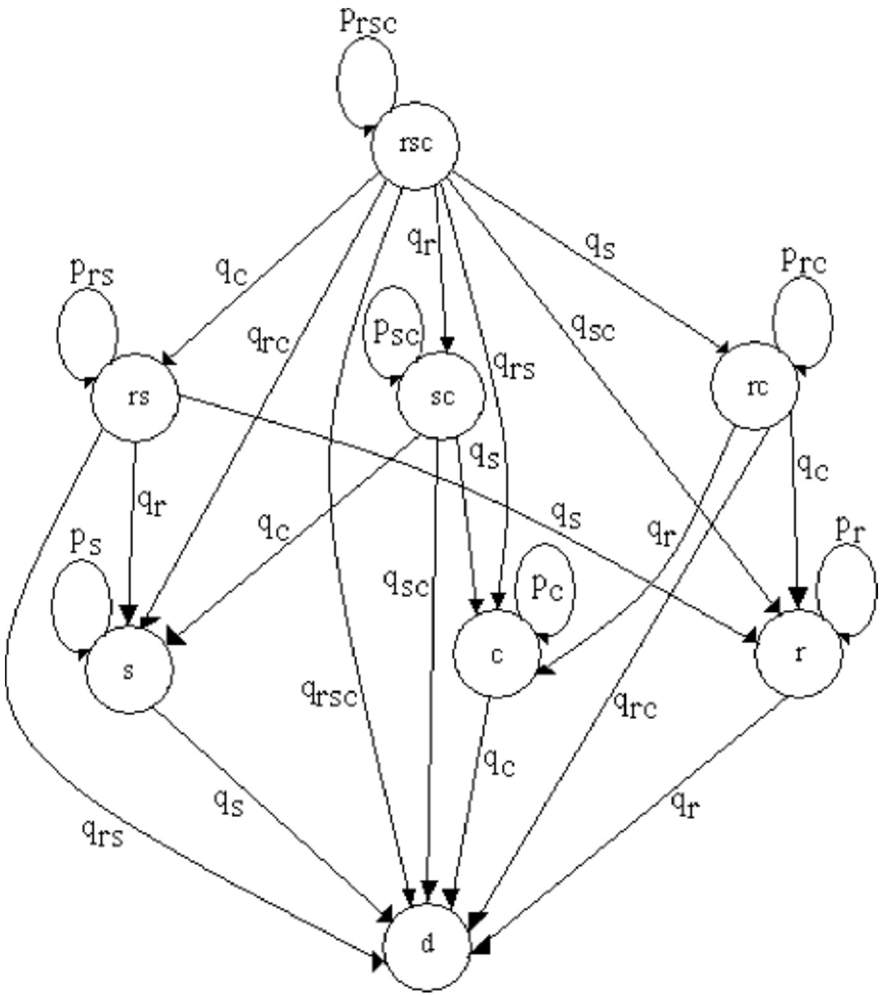

Figure 6.2: three-life annuity graph

In the non-homogeneous case, the MC matrix structure is similar and the only difference is that the non-zero elements will change because of time.
As before we do not enter into the details for the construction of $F_{ij}(t)$ and $F_{ij}(s,t)$.

We suppose that permanence rewards will be variable in time and will be paid at the end of the period and also that the intensity of the interest rate is fixed.
Now we can write the related CTHSMRW and CTNHSMRW evolution equations:

$$V_1(t)=\left(1-H_1(t)\right)\int_0^t \psi_1(\theta)e^{-\delta\theta}d\theta + \sum_{k=1}^{8}\int_0^t \dot{Q}_{1k}(\vartheta)\int_0^{\vartheta}\psi_1(\theta)e^{-\delta\theta}d\theta d\vartheta + \sum_{k=1}^{7}\int_0^t \dot{Q}_{1k}(\vartheta)e^{-\delta\vartheta}V_k(t-\vartheta)d\vartheta, \tag{6.9}$$

$$V_1(s,t)=\left(1-H_1(s,t)\right)\int_s^t \psi_1(s,\theta)e^{-\delta(\theta-s)}d\theta + \sum_{k=1}^{8}\int_s^t \dot{Q}_{1k}(\vartheta)\int_s^{\vartheta}\psi_1(s,\theta)e^{-\delta(\theta-s)}d\theta d\vartheta + \sum_{k=1}^{7}\int_s^t \dot{Q}_{1k}(s,\vartheta)e^{-\delta(\vartheta-s)}V_k(\vartheta,t)d\vartheta, \tag{6.10}$$

$$V_2(t)=\left(1-H_2(t)\right)\int_0^t \psi_2(\theta)e^{-\delta\theta}d\theta + \sum_{k=2}^{8}\int_0^t \dot{Q}_{2k}(\vartheta)\int_0^{\vartheta}\psi_2(\theta)e^{-\delta\theta}d\theta d\vartheta + \sum_{k=2}^{7}\int_0^t \dot{Q}_{2k}(\vartheta)e^{-\delta\vartheta}V_k(t-\vartheta)d\vartheta, \tag{6.11}$$

$$V_2(s,t)=\left(1-H_2(s,t)\right)\int_s^t \psi_2(s,\theta)e^{-\delta(\theta-s)}d\theta + \sum_{k=2}^{8}\int_s^t \dot{Q}_{2k}(s,\vartheta)\int_s^{\vartheta}\psi_2(s,\theta)e^{-\delta(\theta-s)}d\theta d\vartheta + \sum_{k=2}^{7}\int_s^t \dot{Q}_{2k}(s,\vartheta)e^{-\delta(\vartheta-s)}V_k(\vartheta,t)d\vartheta, \tag{6.12}$$

$$V_3(t)=\left(1-H_3(t)\right)\int_0^t \psi_3(\theta)e^{-\delta\theta}d\theta + \sum_{k=3}^{8}\int_0^t \dot{Q}_{3k}(\vartheta)\int_0^{\vartheta}\psi_3(\theta)e^{-\delta\theta}d\theta d\vartheta + \sum_{k=3}^{7}\int_0^t \dot{Q}_{3k}(\vartheta)e^{-\delta\vartheta}V_k(t-\vartheta)d\vartheta, \tag{6.13}$$

$$V_3(s,t)=\left(1-H_3(s,t)\right)\int_s^t \psi_3(s,\theta)e^{-\delta(\theta-s)}d\theta + \sum_{k=3}^{8}\int_s^t \dot{Q}_{3k}(s,\vartheta)\int_s^{\vartheta}\psi_3(s,\theta)e^{-\delta(\theta-s)}d\theta d\vartheta + \sum_{k=3}^{7}\int_s^t \dot{Q}_{3k}(s,\vartheta)e^{-\delta(\vartheta-s)}V_k(\vartheta,t)d\vartheta, \tag{6.14}$$

$$V_4(t)=\left(1-H_4(t)\right)\int_0^t \psi_4(\theta)e^{-\delta\theta}d\theta + \sum_{k=4}^{8}\int_0^t \dot{Q}_{4k}(\vartheta)\int_0^{\vartheta}\psi_4(\theta)e^{-\delta\theta}d\theta d\vartheta + \sum_{k=4}^{7}\int_0^t \dot{Q}_{4k}(\vartheta)e^{-\delta\vartheta}V_k(t-\vartheta)d\vartheta, \tag{6.15}$$

$$V_4(s,t) = (1 - H_4(s,t)) \int_s^t \psi_4(s,\theta) e^{-\delta(\theta-s)} d\theta$$
$$+ \sum_{k=4}^{8} \int_s^t \dot{Q}_{4k}(s,\vartheta) \int_s^{\vartheta} \psi_4(s,\theta) e^{-\delta(\theta-s)} d\theta d\vartheta + \sum_{k=4}^{7} \int_s^t \dot{Q}_{4k}(s,\vartheta) e^{-\delta(\vartheta-s)} V_k(\vartheta,t) d\vartheta, \tag{6.16}$$

$$V_5(t) = (1 - H_5(t)) \int_0^t \psi_5(\theta) e^{-\delta\theta} d\theta$$
$$+ \sum_{k=5}^{8} \int_0^t \dot{Q}_{5k}(\vartheta) \int_0^{\vartheta} \psi_5(\theta) e^{-\delta\theta} d\theta d\vartheta + \int_0^t \dot{Q}_{55}(\vartheta) e^{-\delta\vartheta} V_5(t-\vartheta) d\vartheta, \tag{6.17}$$

$$V_5(s,t) = (1 - H_5(s,t)) \int_s^t \psi_5(s,\theta) e^{-\delta(\theta-s)} d\theta$$
$$+ \sum_{k=5}^{8} \int_s^t \dot{Q}_{5k}(s,\vartheta) \int_s^{\vartheta} \psi_5(s,\theta) e^{-\delta(\theta-s)} d\theta d\vartheta + \int_s^t \dot{Q}_{55}(s,\vartheta) e^{-\delta(\vartheta-s)} V_5(\vartheta,t) d\vartheta, \tag{6.18}$$

$$V_6(t) = (1 - H_6(t)) \int_0^t \psi_6(\theta) e^{-\delta\theta} d\theta$$
$$+ \sum_{k=6}^{8} \int_0^t \dot{Q}_{6k}(\vartheta) \int_0^{\vartheta} \psi_6(\theta) e^{-\delta\theta} d\theta d\vartheta + \int_0^t \dot{Q}_{66}(\vartheta) e^{-\delta\vartheta} V_6(t-\vartheta) d\vartheta, \tag{6.19}$$

$$V_6(s,t) = (1 - H_6(s,t)) \int_s^t \psi_6(s,\theta) e^{-\delta(\theta-s)} d\theta$$
$$+ \sum_{k=6}^{8} \int_s^t \dot{Q}_{6k}(s,\vartheta) \int_s^{\vartheta} \psi_6(s,\theta) e^{-\delta(\theta-s)} d\theta d\vartheta + \int_s^t \dot{Q}_{66}(s,\vartheta) e^{-\delta(\vartheta-s)} V_6(\vartheta,t) d\vartheta, \tag{6.20}$$

$$V_7(t) = (1 - H_7(t)) \int_0^t \psi_7(\theta) e^{-\delta\theta} d\theta$$
$$+ \sum_{k=7}^{8} \int_0^t \dot{Q}_{7k}(\vartheta) \int_0^{\vartheta} \psi_7(\theta) e^{-\delta\theta} d\theta d\vartheta + \int_0^t \dot{Q}_{77}(\vartheta) e^{-\delta\vartheta} V_7(t-\vartheta) d\vartheta, \quad . \tag{6.21}$$

$$V_7(s,t) = (1 - H_7(s,t)) \int_s^t \psi_7(s,\theta) e^{-\delta(\theta-s)} d\theta$$
$$+ \sum_{k=7}^{8} \int_s^t \dot{Q}_{7k}(s,\vartheta) \int_s^{\vartheta} \psi_7(s,\theta) e^{-\delta(\theta-s)} d\theta d\vartheta + \int_s^t \dot{Q}_{77}(s,\vartheta) e^{-\delta(\vartheta-s)} V_7(\vartheta,t) d\vartheta. \tag{6.22}$$

From the graph depicted in **Figure 6.2**, each node constitutes an equivalence class and the equivalence classes are:

$C_1 = \{rsc\}$, $C_2 = \{rs\}$, $C_3 = \{sc\}$, $C_4 = \{rc\}$, $C_5 = \{r\}$, $C_6 = \{s\}$, $C_7 = \{c\}$, $C_8 = \{d\}$.
Class C_1 is the only maximal class and C_8 the only minimal (absorbing, essential) class and all the other classes are transient.
For these reasons once the system is in a state, it cannot come back.
In the formulas the sum starts always from the initial state.
For obvious reasons, the absorbing state does not make any contribution to the rewards, and is not considered in the last part of each formula.
In the last six formulas it results that for the starting state the only real transition is with state 8 (death state). The formulas take into account this property and the last element takes into account the only transition that is possible, the virtual one.

7 INSURANCE MODEL WITH STOCHASTIC INTEREST RATES

7.1 Introduction

In this section we apply homogeneous or non-homogeneous semi-Markov rewards to the actuarial field using a stochastic term structure of implied forward rates.
To apply the model, first it is necessary to solve the semi-Markov evolution equation to get the stochastic interest rates problem as presented in section 3 and then we have to solve the related homogeneous semi-Markov reward process.
In the homogeneous reward case the semi-Markov interest model should be homogeneous; however in the non-homogeneous case we can have a homogeneous or non-homogeneous semi-Markov interest rate model.
In the non-homogeneous case only the non-homogeneous interest rate model will be presented.
The reward process will be extended with the introduction of stochastic interest rates.

7.2 The Actuarial Problem

We now consider a model with *m* states presented in the graph of **Figure 7.1**.
It must be made clear that the arcs depict all possible transitions; they are weighted and their weights represent the transition probabilities and the rewards that are paid in the case of transition on the arc.
The weights will be represented by a pair (*p,r*) where *p* represents the probability and *r* the related reward, which can be a positive or negative value, depending on whether it is respectively an entrance or a payment.
If *p* is equal to 0, then *r* must be 0 because if it is not possible to cross the arc the related reward is meaningless.

Furthermore, the nodes representing the model states are also weighted and their weights represent the permanence reward paid or received remaining in the considered state.

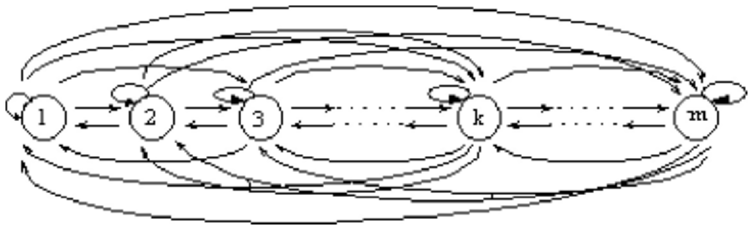

Figure 7.1: *m* states model for insurance models.

All rewards can be fixed or can change during the time evolution of the model.
The different models can be constructed giving different values for *p*. For example if a node has all the probabilities of leaving the related state equal to 0, then this will be an absorbing state.

7.3 A Semi-Markov Reward Stochastic Interest Rate Model

A stochastic term structure of implied forward rates can be constructed by means of SMP in both homogeneous and non-homogeneous cases, as explained before.
As usual the evolution equation of the DTSMP will be the following one:

$$\phi_{ij}(t) = \delta_{ij}(1-H_i(t)) + \sum_{\beta=1}^{n}\sum_{\vartheta=1}^{t}\phi_{\beta j}(t-\vartheta)b_{i\beta}(\vartheta), \tag{7.1}$$

$$\phi_{ij}(s,t) = \delta_{ij}(1-H_i(s,t)) + \sum_{\beta=1}^{n}\sum_{\vartheta=s+1}^{t}\phi_{\beta j}(\vartheta,t)b_{i\beta}(s,\vartheta), \tag{7.2}$$

where $\phi_{ij}(t)$ and $\phi_{ij}(s,t)$ represent, as already said, the probability that at time t the implied interest rate will be r_j given that the implied interest rate was r_i at time 0 in the homogeneous case and s in the non-homogeneous. We suppose that the states of interest rate model are:

$$E = \{r_1, r_2, \ldots, r_n\}. \tag{7.3}$$

The related mean discount factors at time h are constructed as explained in section 3.
More precisely $\nu_\varepsilon(h)$ and $\nu_\varepsilon(s,h)$ represent the mean discounting factor for a time h given that at epoch 0 (s) the interest rate was r_ε.
The states of the reward process are always

$$I = \{S_1, S_2, \ldots, S_m\} \tag{7.4}$$

and the general evolution equations of homogeneous and non-homogeneous reward processes in the immediate case are respectively given by:

$$V_i^{\varepsilon}(t) = \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\sum_{\tau=1}^{\vartheta}\psi_{ik}(\tau)\nu_{\varepsilon}(\tau) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\gamma_{ik}(\vartheta)\nu_{\varepsilon}(\vartheta)$$
$$+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\overline{V}_k^{\varepsilon}(t-\vartheta)\nu_{\varepsilon}(\vartheta) + (1-swpe)(1-H_i(t))\sum_{\tau=1}^{t}\psi_i(\tau)\nu_{\varepsilon}(\tau) \quad (7.5)$$
$$+swpe(1-H_i(t))\sum_{k=1}^{m}\sum_{\tau=1}^{t}\varphi_{ik}(t)\psi_{ik}(\tau)\nu_{\varepsilon}(\tau),$$

$$V_i^{\varepsilon}(s,t) = swpe(1-H_i(s,t))\sum_{k=1}^{m}\sum_{\tau=s+1}^{t}\varphi_{ik}(s,t)\psi_{ik}(s,\tau)\nu_{\varepsilon}(s,\tau)$$
$$+(1-swpe)(1-H_i(s,t))\sum_{\tau=s+1}^{t}\psi_i(s,\tau)\nu_{\varepsilon}(s,\tau)$$
$$+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\overline{V}_k^{\varepsilon}(\vartheta,t)\nu_{\varepsilon}(s,\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta)\nu_{\varepsilon}(s,\vartheta) \quad (7.6)$$
$$+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\sum_{\tau=s+1}^{\vartheta}\psi_{ik}(s,\tau)\nu_{\varepsilon}(s,\tau),$$

where:

$$\overline{V}_{\beta}^{\varepsilon}(t-\theta) = \sum_{j=1}^{n}\phi_{\varepsilon j}(\theta)V_{\beta}^{j}(t-\theta), \quad (7.7)$$

$$\overline{V}_{\beta}^{\varepsilon}(\theta,t) = \sum_{j=1}^{n}\phi_{\varepsilon j}(s,\theta)V_{\beta}^{j}(\theta,t), \quad (7.8)$$

given that r_{ε} was the implied interest rate at time 0.

Let us point out that in relation (7.7), the term $V_{\beta}^{j}(t-\theta)$ represents the RMPV of all the rewards paid or received in a time $t-\theta$, given that the system is in state S_{β} and the interest rate is r_j.

At starting time 0 of our study, the *known interest rate* was r_{ε}. The system evolves for a time θ and gets the interest r_j with the probability $\phi_{\varepsilon j}(\theta)$.

To compute the mean RMPV, we first need to compute the expected value (7.7) and then to use the general formula.

Similar arguments can be used for relation (7.8) in the non-homogeneous case.

Furthermore if we don't know the interest rate at the initial moment, it results that:

$$V_i(t) = \sum_{\varepsilon=1}^{n} p_{\varepsilon}V_i^{\varepsilon}(t), \quad (7.9)$$

$$V_i(s,t) = \sum_{\varepsilon=1}^{n} p_\varepsilon(s) V_i^\varepsilon(s,t), \tag{7.10}$$

where:

$$p_1, p_2, \ldots, p_n \tag{7.11}$$

and

$$p_1(s), p_2(s), \ldots, p_n(s) \tag{7.12}$$

are the initial probability distributions of the r.v. interest rate respectively at time 0 in the homogeneous case and at time s in the non-homogeneous one.
In the non-homogeneous case if (7.12) is unknown, we can work in the following way:

$$p_\varepsilon(0,s) = \sum_{\varepsilon=1}^{n} p_i(0,0)\phi_{i\varepsilon}(0,s) \tag{7.13}$$

where $p_i(0,0)$ are the elements of initial probability distribution and should be known.
The related due cases are given by:

$$\begin{aligned}\ddot{V}_i^\varepsilon(t) &= \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\sum_{\tau=0}^{\vartheta-1}\psi_{ik}(\tau)\nu_\varepsilon(\tau) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\gamma_{ik}(\vartheta)\nu_\varepsilon(\vartheta) \\ &+ \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\ddot{\bar{V}}_k^\varepsilon(t-\vartheta)\nu_\varepsilon(\vartheta) + (1-swpe)(1-H_i(t))\sum_{\tau=0}^{t-1}\psi_i(\tau)\nu_\varepsilon(\tau) \\ &+ swpe(1-H_i(t))\sum_{k=1}^{m}\sum_{\tau=0}^{t-1}\varphi_{ik}(t)\psi_{ik}(\tau)\nu_\varepsilon(\tau),\end{aligned} \tag{7.14}$$

$$\begin{aligned}\ddot{V}_i^\varepsilon(s,t) &= swpe(1-H_i(s,t))\sum_{k=1}^{m}\sum_{\tau=s}^{t-1}\varphi_{ik}(s,t)\psi_{ik}(s,\tau)\nu_\varepsilon(s,\tau) \\ &+ \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\sum_{\tau=s}^{\vartheta-1}\psi_{ik}(s,\tau)\nu_\varepsilon(s,\tau) \\ &+ (1-swpe)(1-H_i(s,t))\sum_{\tau=s}^{t-1}\psi_i(s,\tau)\nu_\varepsilon(s,\tau) + \\ &\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta)\nu_\varepsilon(s,\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\ddot{\bar{V}}_k^\varepsilon(\vartheta,t)\nu_\varepsilon(s,\vartheta).\end{aligned} \tag{7.15}$$

In the due case, equations (7.7) and (7.8) become:

$$\ddot{\bar{V}}_\beta^\varepsilon(t-\theta) = \sum_{j=1}^{n}\phi_{\varepsilon j}(\theta)\ddot{V}_\beta^j(t-\theta), \tag{7.16}$$

$$\ddot{\bar{V}}_\beta^\varepsilon(\theta,t) = \sum_{j=1}^{n}\phi_{\varepsilon j}(s,\theta)\ddot{V}_\beta^j(\theta,t). \tag{7.17}$$

At last it results that:

$$\ddot{V}_i(t) = \sum_{\varepsilon=1}^{n} p_\varepsilon \ddot{V}_i^\varepsilon(t), \tag{7.18}$$

$$\ddot{V}_i(s,t) = \sum_{\varepsilon=1}^{n} p_\varepsilon(s) \ddot{V}_i^\varepsilon(s,t). \tag{7.19}$$

To apply this model we should, in both immediate and due cases, solve the two different evolution equations, the first for the rate of interest and the other one for the reward process. These equations are obtained with two independent data sets. Furthermore, in the solution of the two different evolution equations it is supposed that the two phenomena are independent. But this is not a strong hypothesis because it is obvious that the interest rate structure doesn't depend upon the evolution of an illness.

In conclusion, the semi-Markov reward models presented in this last section are quite general and can take into account two random evolutions, one for the interest rate and the other for the evolution of the actuarial phenomena and constitute strong models for real life applications.

Chapter 7

INSURANCE RISK MODELS

In this chapter, we will first recall the main classical models in *risk theory* which are useful for insurance companies and then extend them fully to the *semi-Markov case.* To avoid confusion we adopt the classical actuarial notation of risk theory

1 CLASSICAL STOCHASTIC MODELS FOR RISK THEORY AND RUIN PROBABILITY

In this section, we will develop **Example 4.1** of Chapter 3, first into a general case, and then into the particular case of a Poisson process for claim arrivals.
Let us consider an insurance company, beginning at time 0 with an initial capital of amount u ($u > 0$), also called *reserve* for insurance companies or *equity* for banks.
In almost all developed countries, this initial reserve has a minimal amount fixed by the government and depending on the turnover of the insurance company. Indeed, it is clearly understood that this capital protects customers against the possibility that an unlucky company would have to pay a lot of large claims in a short period of time, for example for a catastrophic event, and not be liquid enough to do so.
A basic problem, in general solved by actuaries, is to give an objective value for this minimal reserve. We will learn later how to solve this fundamental problem.
Any *risk model* related to an insurance company is characterized by three "basic" processes:

(i) the first one is the *claim number process.* This is a stochastic process giving the counting process of claims occurring to the customers;
(ii) the second stochastic process concerns the *claim amounts*. In particular, it gives the distribution of what the company has to pay when a claim occurs;
(iii) the last process is related to the *income* of the company; and it is generally a deterministic process since the *premiums* paid by the customers must be known at the origin of the individual contracts.

To any set of assumptions about these three processes, there corresponds a particular stochastic *risk model*. The most important will be presented later.
This section will only be concerned with two models: the so-called *G/G model* or the E.S. Andersen model, and the *P/G model* or the Poisson or Cramer-Lundberg model. The notation, borrowed from queuing theory, gives information

concerning the two d.f. used in these models, one for the interarrival and the other one for the claim amounts (G for general means any d.f. and P, for Poisson, a negative exponential distribution

1.1 The G/G Or E.S. Andersen Risk Model

1.1.1 The Model

The basic assumptions for the G/G model are:

(i) *The claim number process*
Let $(X_n, n \geq 1)$ represent the stochastic process of interarrival times between successive claims. We will suppose that this process is a sequence of i.i.d. non - negative random variables with A as common d.f., such that:
a) $A(0) < 1,$ (1.1)
b) $\int_0^\infty x dA(x) = \alpha < \infty.$ (1.2)

(ii) *The claim amount process*
Let $(Y_n, n \geq 1)$ represent the sequence of successive claim amounts. Here too, we will suppose that we have a random sequence of non-negative i.i.d. random variables with B as common d.f., such that:
a) $B(0) < 1,$ (1.3)
b) $\int_0^\infty y dB(y) = \beta < \infty.$ (1.4)
Moreover, the sequences $(X_n, n \geq 1)$ and $(Y_n, n \geq 1)$ are independent and defined on a complete probability space $(\Omega, \Im, P)$.

(iii) *The premium income process*
The classical assumption is that there is a constant, of course positive, *premium rate c* per unit of time, which means that in the time period $[0,t]$, the total amount of income for the company is ct.

1.1.2 The Premium

One of the major problems for the company is how to fix the premium rate "fairly" while respecting two conditions:

a) the *lifetime* of the company, that is the period in which its capital is always positive, must be, with a very large probability, as long as possible.
Indeed from the economic point of view, large reserves constitute a factor of security but excessive reserves may signify that the premiums are too high.

b) It is in the best interest of each company to choose c as low as possible but without infringing upon its own economic security.

To fix the value of c, let us consider the renewal process $(T_n, n \geq 0)$ of claim arrival times related to the sequence $(X_n, n \geq 1)$, with X_0=0 a.s.. That is:

$$T_n = \sum_{k=0}^{n} X_k. \tag{1.5}$$

Using renewal theory, the associated counting process $(N(t), t \geq 0)$ defined by (2.3) of Chapter 2, gives the total number of claims in (0,t] and from **Corollary 4.2**, Chapter 2, relation (4.26), we know that:

$$\lim_{t \to \infty} \frac{H(t)}{t} = \frac{1}{\alpha} \tag{1.6}$$

if

$$H(t) = E(N(t)) \tag{1.7}$$

and so, for large t:

$$E(N(t)) \approx \frac{t}{\alpha}. \tag{1.8}$$

Now, from relation (1.4) the mean cost of the total number of claims in (0,t] is approximately:

$$\frac{\beta}{\alpha} t. \tag{1.9}$$

This last result shows that the mean total cost of claims having to be paid by the insurance company during the period (0,t] is approximately given by $\tilde{c}t$, where:

$$\tilde{c} = \frac{\beta}{\alpha}. \tag{1.10}$$

It follows that if we take this value $\tilde{c}$ as the constant premium rate per unit of time, we have what is called a game, "insurance company-customers" which is asymptotically fair. That is why $\tilde{c}$ is called the *pure premium*.
But, unfortunately, we will see later that this choice leads to the ruin of the company a.s. on $[0, \infty)$. Also, it is necessary to introduce a positive *security loading* η such that:

$$c = (1 + \eta)\tilde{c} \tag{1.11}$$

or

$$c = (1 + \eta)\frac{\beta}{\alpha}. \tag{1.12}$$

In other words, the company must choose the premium rate c such that:

$$c > \frac{\beta}{\alpha}. \tag{1.13}$$

Now, $\tilde{c}$ is called the *loading premium*.

So, if we set c = 1, this relation implies the assumption:

$$\alpha > \beta \,, \tag{1.14}$$

i.e., the mean interarrival time between two successive claims is larger than the mean claim amount.
Intuitively, this condition warrants good service to the policy holders and it will be theoretically justified below.
In conclusion, we can state that each insurance company has to *control* two basic parameters: its *initial reserve or equities u* and its *security loading* η. Moreover, the possibilities open to the insurance companies are dictated by law.

1.1.3 Three Basic Processes

We will now introduce three stochastic processes of fundamental importance in risk theory.

1) *The accumulated claim amount process*
It is the stochastic process $(U(t), t \geq 0)$ defined as:

$$U(t) = \sum_{n=1}^{N(t)} Y_n \tag{1.15}$$

or as:

$$U(t) = U_{N(t)} \tag{1.16}$$

if

$$U_n = \sum_{i=1}^{n} Y_i , \tag{1.17}$$

always using the classical convention that the value of a sum over a void set is zero.
For every fixed t, $U(t)$ gives the total number of claims on $(0,t]$.
Let us denote by $M(t,y)$ the value of the d.f. of $U(t)$ at y; we can then write:

$$M(t,y) = \sum_{n=0}^{\infty} P(U_n \leq y, N(t) = n) \,. \tag{1.18}$$

Using relation (3.5) of Chapter 2, the independence assumption of the two stochastic processes $(X_n, n \geq 1)$ and $(Y_n, n \geq 1)$ leads to:

$$\begin{aligned} M(t,y) &= \sum_{n=0}^{\infty} P(U_n \leq y) P(N(t) = n) \\ &= \sum_{n=0}^{\infty} (A^{(n)}(t) - A^{(n+1)}(t)) B^{(n)}(y). \end{aligned} \tag{1.19}$$

2) *The risk process*
It is the stochastic process:

$$(U(t) - ct, t \geq 0) \tag{1.20}$$

representing the total net outcome of the company up to time *t*, provided it is still alive at this time.

3) *The risk reserve process (or the surplus process)*
It represents the stochastic process $(\alpha(t), t \geq 0)$, where:

$$\alpha(t) = u - U(t) + ct, t \geq 0. \tag{1.21}$$

It gives, at time *t*, the total net asset of the company supposing the company is still alive at time *t*.
The next two figures give typical trajectories of the *N* process and the α process.

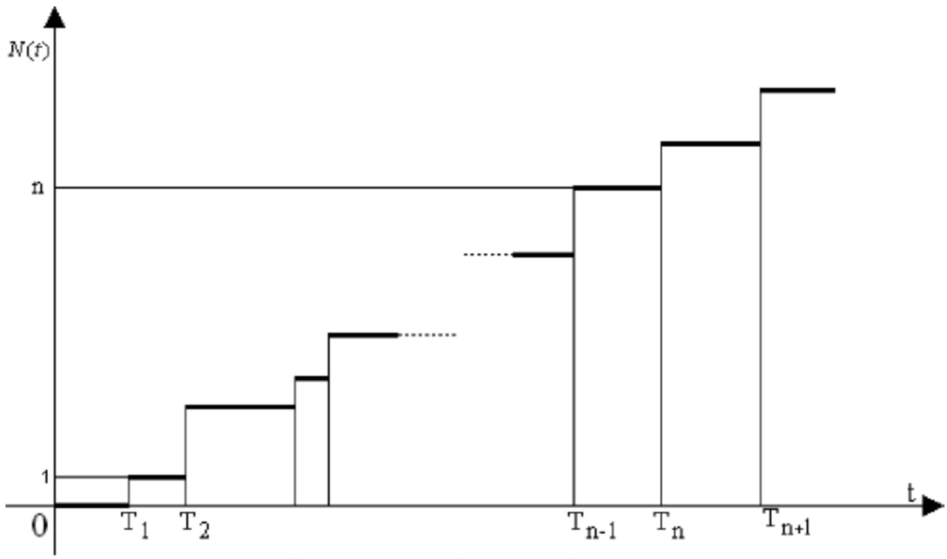

Figure 1.1: trajectory of *N* process

1.1.4 The Ruin Problem

We come now to the fundamental *ruin problem* in risk theory.
From a strict economic point of view, the lifetime of the insurance company may be defined as the stopping time:

$$T = \inf\{t : \alpha(t) < 0\}. \tag{1.22}$$

This is a "strict" point of view, as we do not consider the possibility for the company to take out a loan to cover a "small" ruin.
Clearly, if the event $\{\omega : T(\omega) \leq 0\}$ occurs, then the company is ruined before or at time *t*; otherwise the company is still alive at time *t*.

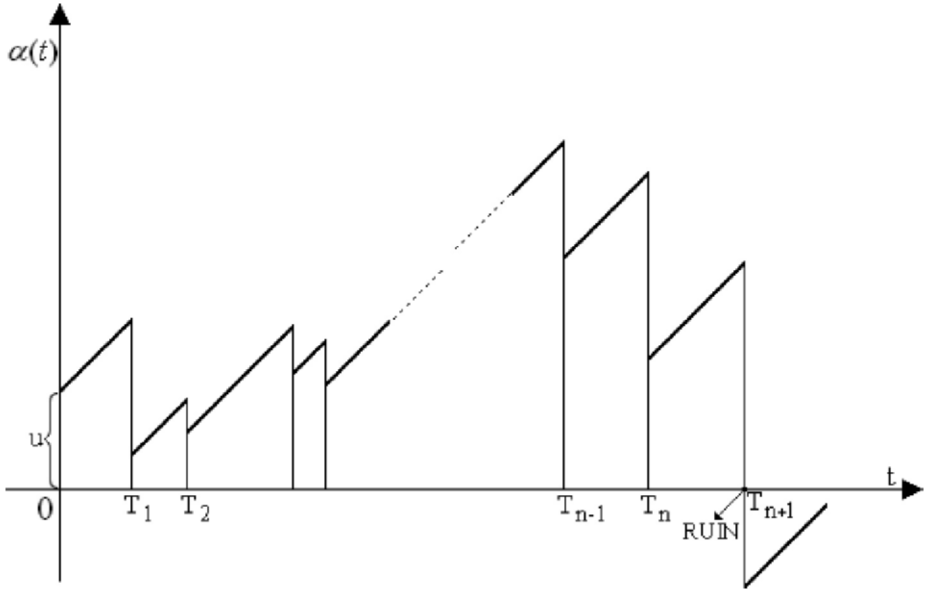

Figure 1.2: trajectory of α process

We will use the following notation for the *probabilities of ruin and non-ruin on an infinite time horizon*, i.e. on $[0,\infty)$:

$$\Psi(u) = P(T < \infty \mid \alpha(0) = u), \tag{1.23}$$

$$\phi(u) = P(T = \infty \mid \alpha(0) = u) = 1 - \Psi(u). \tag{1.24}$$

The knowledge of Ψ or equivalently of ϕ is necessary in order to select values for parameters u and for η to warrant good services for the customers.
For example, if u is fixed, we can see the probability ϕ as a function of the security loading η, say:

$$\phi(u,\eta). \tag{1.25}$$

If we impose the condition:

$$\phi(u,\eta) > \varepsilon, \tag{1.26}$$

for example with ε =0.99999, we can select the minimum value of η such that condition (1.26) is satisfied.
With the aid of results on random walk, we can now justify theoretically the fact that a strictly positive security loading is a necessary condition for not having ruin on $[0,\infty)$ a.s.
On the time period $(T_{n-1},T_n]$, the liability of the company increases or decreases by a net amount given by:

$$Z_n = Y_n - cX_n, n \geq 1. \tag{1.27}$$

The sequence of i.i.d. r.v.

$$(Z_n, n \geq 1) \tag{1.28}$$

generates a random walk of successive values:

$$S_n = \sum_{k=1}^{n} Z_k. \tag{1.29}$$

From relation (1.21), we get:

$$\alpha(T_n) = u - S_n \tag{1.30}$$

since S_n is the risk process value at time T_n.
Let us now consider the r.v. M defined by relation (17.27) of Chapter 3; from relation (1.24), we deduce that:

$$\phi(u) = P(M \leq u). \tag{1.31}$$

From **Proposition 17.1** of Chapter 3, we know that the d.f. of M is non-degenerate iff the random walk drifts toward $-\infty$, or equivalently iff:

$$E(Z_n) < 0. \tag{1.32}$$

It is now clear that this last condition is, from relation (1.27), also equivalent to the inequality (1.13).
The case

$$\beta - c\alpha = 0 \tag{1.33}$$

must be treated carefully.
Indeed, in this case, the random walk generated by the random sequence (1.27) oscillates, so that for any positive u, we have:

$$P(\exists n \in \mathbb{N}_0 : S_n > u) = 1. \tag{1.34}$$

In other words, this result shows that whatever the initial reserve is, the company will be ruined with probability 1. This also means that the asymptotic fair game leads a.s. to the ruin of the company.
So, without any loading, the random walk $(S_n, n \geq 0)$ will either drift toward $+\infty$ or oscillate. In both cases, we know that a.s.

$$M = \infty. \tag{1.35}$$

It follows that the problem of computing the non-ruin probability function ϕ only arises when inequalities (1.13) or (1.14) are satisfied, and it is necessary to particularize some basic assumptions in order to obtain more tractable analytical expressions. This is possible in the case of the *P/G* or Cramer-Lundberg model.

1.2 The P/G Or CRAMER-LUNDBERG Risk Model

1.2.1 The Model

To obtain this particular risk model, it suffices to adapt the Andersen model explained above in the following manner: we impose that the claim amount process is a Poisson process or as in **Example 3.1**, Chapter 2 that:

$$A(x)=\begin{cases}1-e^{-\lambda x}, x\geq 0,\\ 0, \qquad x<0,\end{cases} \tag{1.36}$$

so that, by relation (1.2):

$$\alpha=\frac{1}{\lambda}. \tag{1.37}$$

Condition (1.13) or (1.32) becomes:

$$c>\lambda\beta. \tag{1.38}$$

So, if in general, any Andersen model is defined by two general d.f. A and B on $[0,\infty)$, which, as we already know, justifies the notation G/G model where letter G stands for "general", on the other hand, any Cramer-Lundberg model is defined by a strictly positive parameter λ, defining the Poisson process of claim arrivals and by a general d.f. B on $[0,\infty)$ for claim amounts. This also explains the notation P/G (P for "Poisson" and G for "general") for this particular model.

1.2.2 The Ruin Probability

Now we will see how it is possible to build specific mathematical treatments to obtain simple results concerning the non-ruin probability function ϕ.

From now on, we will suppose that condition (1.38) is satisfied; otherwise, ϕ is identically 0.

From standard rules of probability, we get, by conditioning with respect to the first claim occurrence time,

$$\phi(u)=\int_0^\infty \lambda e^{-\lambda t}\int_0^{u+ct}\phi(u+ct-y)dB(y)dt, u>0. \tag{1.39}$$

By the change of variables $z=u+ct$, we get:

$$\phi(u)=\frac{\lambda}{c}e^{\frac{\lambda}{c}u}\int_c^\infty e^{-\frac{\lambda}{c}z}\int_0^z\phi(z-y)dB(y)dz. \tag{1.40}$$

From classical theorems of analysis, it follows from this last expression that ϕ is derivable and that its derivative can be computed as follows:

$$\begin{aligned}\phi'(u)=&\left(\frac{\lambda}{c}e^{\frac{\lambda}{c}u}\right)'\int_u^\infty e^{-\frac{\lambda}{c}z}\int_0^z\phi(z-y)dB(y)dz\\&+\frac{\lambda}{c}e^{\frac{\lambda}{c}u}\left(-e^{\frac{\lambda}{c}u}\right)\int_0^u\phi(z-y)dB(y).\end{aligned} \tag{1.41}$$

Using relation (1.40) again, we get:

$$\phi'(u)=\frac{\lambda}{c}\phi(u)-\frac{\lambda}{c}\int_0^u\phi(u-y)dB(y). \tag{1.42}$$

We will now integrate this last equality term by term on $[0,t]$ to obtain:

$$\phi(t)-\phi(0)=\frac{\lambda}{c}\int_0^t \phi(\xi)d\xi-\frac{\lambda}{c}\int_0^t\int_0^{\xi}\phi(\xi-y)dB(y)d\xi. \tag{1.43}$$

Using Fubini's theorem related to the permutation of integration for the last term of the second member of (1.43), we get:

$$\phi(t)-\phi(0)=\frac{\lambda}{c}\int_0^t \phi(\xi)d\xi-\frac{\lambda}{c}\int_0^t\int_t^{y}\phi(\xi-y)dB(y)d\xi. \tag{1.44}$$

The double integral of the second member can be integrated by parts, with:

$$f(y)=\int_0^{t-y}\phi(v)dv(=\int_y^t\phi(\xi-y)d\xi),dg(y)=dB(y). \tag{1.45}$$

This leads to the following result:

$$\phi(t)-\phi(0)=\frac{\lambda}{c}\phi(\xi)d\xi-\frac{\lambda}{c}\int_0^t\phi(t-y)B(y)dy. \tag{1.46}$$

Finally, setting $\xi=t-y$ in the first integral of this last relation, we get:

$$\phi(t)=\phi(0)+\frac{\lambda}{c}\int_0^t\phi(t-y)(1-B(y)dy. \tag{1.47}$$

Before solving this integral equation, we have to compute the value of $\phi(0)$. To do so, we let *t* tend toward ∞ in the last relation; it follows that:

$$\phi(\infty)=\phi(0)+\frac{\lambda}{c}\phi(\infty)L(\infty) \tag{1.48}$$

where: $dL(y)=[1-B(y)]dy$ and so

$$L(\infty)=\int_0^{\infty}(1-B(y))dy=\beta. \tag{1.49}$$

Now, coming back to the equality (1.48), we can extract the value of $\phi(\infty)$:

$$\phi(\infty)=\frac{\phi(0)}{1-\frac{\lambda\beta}{c}}, \tag{1.50}$$

but, by condition (1.38), we have $\phi(\infty)=1$ and so the last relation gives the desired result:

$$\phi(0)=1-\frac{\lambda\beta}{c}. \tag{1.51}$$

The final form of the integral equation (1.39) is thus:

$$\begin{aligned}\phi(t)&=1-\frac{\lambda\beta}{c}+\frac{\lambda\beta}{c}\int_0^t\phi(t-y)dB^*(y),\\ B^*(y)&=\frac{1}{\beta}\int_0^y(1-B(z))dz.\end{aligned} \tag{1.52}$$

We can easily solve this integral equation using the Laplace transform. With the same conventions as in Chapter 2, relation (1.52) leads to

$$\tilde{\phi}(s)=\left(1-\frac{\lambda\beta}{c}\right)\frac{1}{s}+\frac{\lambda\beta}{c}\tilde{\phi}(s)\tilde{b}^*(s) \tag{1.53}$$

where:

$$b^*(y)=\frac{1-B(y)}{\beta}. \tag{1.54}$$

From the algebraic equation (1.53) we can obtain an explicit expression of the Laplace transform of the probability of non-ruin ϕ:

$$\tilde{\phi}(s)=\frac{\left(1-\frac{\lambda\beta}{c}\right)\frac{1}{s}}{\left(1-\frac{\lambda\beta}{c}\tilde{b}^*(s)\right)}. \tag{1.55}$$

Using assumption (1.38), we get

$$1-\frac{\lambda\beta}{c}\tilde{b}^*(s)>1-\frac{\lambda\beta}{c}\tilde{b}^*(0)>1-\frac{\lambda\beta}{c}>0,\forall s>0, \tag{1.56}$$

so that

$$\frac{\lambda\beta}{c}\tilde{b}^*(s)<1,\forall s>0. \tag{1.57}$$

By a series expansion, the new expression of the Laplace transform of ϕ is:

$$\tilde{\phi}(s)=\left(1-\frac{\lambda\beta}{c}\right)\frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{\lambda\beta}{c}\tilde{b}^*(s)\right)^n. \tag{1.58}$$

By inverting this last relation member by member, we get an explicit form of the non-ruin probability:

$$\phi(u)=\left(1-\frac{\lambda\beta}{c}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda\beta}{c}\right)^n B^*(u)^{(n)}. \tag{1.59}$$

If we want to express the probability of ruin at u, we have from equality (1.24):

$$\begin{aligned}\Psi(u)&=\left(1-\frac{\lambda\beta}{c}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda\beta}{c}\right)^n B^*(u))^{(n)},\\&=\left(1-\frac{\lambda\beta}{c}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda\beta}{c}\right)^n (1-B^*(u))^{(n)}.\end{aligned} \tag{1.60}$$

This result was proved by Janssen (1969a).

Example 1.1 *The P/P model or Lundberg's model*
The notation *P/P* means that we must drastically particularize the choice of the d.f. *B* as a negative exponential one so that:

$$B(y)=\begin{cases}1-e^{-\frac{1}{\beta}y}, & y\geq 0,\\ 0, & y<0.\end{cases} \tag{1.61}$$

As here:

$$B^*(y) = B(y), \tag{1.62}$$

we deduce that:

$$\tilde{b}^*(s) = \frac{1}{\beta s + 1} \tag{1.63}$$

and it suffices to use result (1.55) to get:

$$\tilde{\phi}(s) = \frac{\left(1 - \frac{\lambda\beta}{c}\right)\frac{1}{s}}{\left(1 - \frac{\lambda\beta}{c}\right)\frac{1}{\beta s + 1}}. \tag{1.64}$$

Taking the inverse Laplace transform for both members of this last relation, we find that:

$$\phi(u) = 1 - \frac{\lambda\beta}{c} e^{-\left(\frac{1}{\beta} - \frac{\lambda}{c}\right)u} \tag{1.65}$$

and of course:

$$\Psi(u) = \frac{\lambda\beta}{c} e^{-\left(\frac{1}{\beta} - \frac{\lambda}{c}\right)u}. \tag{1.66}$$

The existence of such simple explicit formulas is exceptional in risk theory because it gives very simple expressions for ϕ and Ψ.

Let us also mention that we can write another expression for Ψ; indeed, we know from relation (1.12) that:

$$c = \lambda\beta(1 + \eta). \tag{1.67}$$

Substituting c given by this expression in relation (1.66) leads to the new expression:

$$\Psi(u) = \frac{1}{1 + \eta} e^{-\frac{\eta}{1+\eta}\frac{u}{\beta}}. \tag{1.68}$$

This gives the surprising result that, providing relation (1.38) is satisfied, that is: $c > \lambda\beta$, the ruin probability only depends on η and β but not on λ.

In other words, this result means that if we have two insurance companies having for the *P/P* model respectively the parameters (λ_1, β), (λ_2, β), both couples satisfying inequality (1.38), then these two companies, starting with identical equities, have the *same* ruin probability iff they use the same security loading η

So, in this case, from the point of view of ruin theory, the company having the largest parameter λ is not more dangerous than the other provided that both companies have the same mean claim amount and security loading.

Example 1.2

1) Let us consider an insurance company having as annual mean of claim amounts 2 billions € and 50 000 as mean claim number per year and as initial reserve u an amount of 8 000 000 €.

Using our notation, we see that for the considered company:

$$\begin{aligned} &\lambda = 50000, \\ &\lambda\beta = 2000000000,\ u = 8000000 \end{aligned} \tag{1.69}$$

and so:

$$\beta = 40000. \tag{1.70}$$

From relation (1.68), we get:

$$\Psi(8000000) = \frac{1}{1+\eta} e^{-200\frac{\eta}{\eta+1}}. \tag{1.71}$$

Table 1.1 gives some values of this ruin probability as a function of the loading factor.

loading	*ruin probability*
0.01	0.1366752
0.03	0.0028661
0.05	0.0000696
0.07	0.0000019
0.10	0.000000011544

Table 1.1

2) Now let us suppose that the loading factor is fixed at 7% and let us see what happens if:

(i) the initial reserve has successively the following values: 4 000 000, 2 000 000, 1 000 000, 500 000,

(ii) with a reserve of 8 000 000, the mean claim amount has as values 25 000 and 100 000,

(iii) the annual mean claim number has as values successively 70 000, 20 000, still with a reserve of 8 000 000.

The results are the following:

(i)
$$\begin{aligned} \Psi(2000000) &= \frac{1}{1.07} e^{-\frac{0.07}{1.07}\cdot\frac{u}{40000}}, \\ &= 0.9345794 e^{-0.0654206\frac{u}{40000}} \end{aligned} \tag{1.72}$$

and so we get the results in **Table 1.2**.

reserve	*Ruin probability*
4 000 000	0.0073472
2 000 000	0.0354835
1 000 000	0.1821047
500 000	0. 4125424

Table 1.2

(ii) Here we get:

$$\Psi(8000000) = \frac{1}{1,07} e^{-\frac{0.07}{1.07}\cdot\frac{8000000}{\beta}},$$

$$= 0.9345794 e^{-0.0654206\frac{u}{\beta}}$$

and so we get the results given in **Table 1.3**:

mean claim amount	*ruin probability*
25 000	0. 00000000075655
150 000	0. 0285312

Table 1.3

(iii) In case of an annual mean claim number having as values successively 70 000, 20 000, still with a reserve of 8 000 000, the annual mean of claim amounts has the values 2.8 billions and 800 millions. For the annual mean incomes, we get respectively 996 billions, 856 000 000.
Nevertheless, the ruin probability remains equal to 0.0000019.

1.2.3 Risk Management Using Ruin Probability

The explicit result (1.68) easily gives the solution of the three *basic problems* in risk management for an insurance company.

Problem 1 Given the basic data (λ, β, η) of the company and the initial reserve amount u, how are we to measure the risk exposure of the company?
It suffices to use result (1.68) to compute the ruin probability on $[0, \infty)$.

Problem 2 Given the data (λ, β) of the company and the initial reserve amount u, how are we to measure the loading security so that the ruin probability on $[0, \infty)$ will never exceed a critical value $(1-\varepsilon)$?
Using result (1.68) again, we must solve the inequality:

$$\frac{1}{1+\eta} e^{-\frac{\eta}{1+\eta}\frac{u}{\beta}} < 1-\varepsilon \tag{1.73}$$

or the equality:

$$-\ln(1+\eta) + \frac{\eta}{\eta+1}\frac{u}{\beta} = \ln(1-\varepsilon) \tag{1.74}$$

which can be done using the Newton method.

Problem 3 Given the data (λ, β) of the company and the loading security, how are we to measure the initial reserve amount u so that the ruin probability on $[0,\infty)$ will never exceed a critical value 1-ε?
Using result (1.68) again, we must solve the inequality:

$$\frac{1}{1+\eta} e^{-\frac{\eta}{1+\eta}\frac{u}{\beta}} < 1-\varepsilon \tag{1.75}$$

or the equality:

$$-\ln(1+\eta) + \frac{\eta}{\eta+1}\frac{u}{\beta} = \ln(1-\varepsilon) \tag{1.76}$$

which has as unique solution:

$$\frac{1+\eta}{\eta}\beta \ln(\eta(1+\eta)). \tag{1.77}$$

1.2.4 Cramer's estimator

In the preceding section, we find an explicit expression for the ruin probability ψ using result (1.60). To get a more useful result, from the computational point of view, the only possibility is to obtain simple and good approximations of the function Ψ.
To do this, we start from the integral equation (1.52) and we express the ruin probability Ψ using relation (1.23). We get

$$\Psi(t) = \frac{\lambda\beta}{c}(1 - B^*(t)) + \frac{\lambda\beta}{c}\int_0^t \Psi(t-y)dB^*(y). \tag{1.78}$$

From condition (1.38), we have:

$$\frac{\lambda\beta}{c}\int_0^\infty dB^*(y) < \frac{\lambda\beta}{c} < 1 \tag{1.79}$$

and so it follows that the integral equation (1.78) is not of renewal type; nevertheless, it is said to be of *defective renewal type*.
To be able to apply results of renewal theory from Chapter 2, we have to get around this difficulty; that is why we will pose:

$$\hat{\Psi}(t) = e^{Rt}\Psi(t) \tag{1.80}$$

where R is a positive constant.
Consequently, the integral equation (1.78) can now be written in the form

$$e^{-Rt}\hat{\Psi}(t) = \frac{\lambda\beta}{c}(1 - B^*(t)) + \frac{\lambda\beta}{c}e^{-Rt}\int_0^t e^{-Rt}\hat{\Psi}(t - ydB^*(y)). \tag{1.81}$$

Multiplying both members of this last equality by e^{Rt}, we get:

$$\hat{\Psi}(t) = \frac{\lambda\beta}{c}(1 - B^*(t))e^{Rt} + \frac{\lambda\beta}{c}\int_0^t e^{-Rt}\hat{\Psi}(t - ydB^*(y)) . \tag{1.82}$$

This last equation will be of renewal type provided that

$$\frac{\lambda\beta}{c}\int_0^\infty e^{Ry} dB^*(y) = 1, \tag{1.83}$$

i.e., by relation (1.52), iff the function from $[0,\infty) \mapsto \mathbb{R}^+$,

$$y \mapsto \frac{\lambda}{c} e^{Ry}(1 - B(y)), \tag{1.84}$$

is a density function.
Now using the key renewal theorem (**Proposition 4.2** of Chapter 2), we are able to prove the following fundamental result.

Proposition 1.1 *(Cramer's estimator of ruin theory)*
If:

(i) $\frac{\lambda\beta}{c} < 1,$ (1.85)

(ii) *there exists a positive constant R such that:*

$$\frac{\lambda\beta}{c}\int_0^\infty e^{Ry} dB^*(y) = 1 \tag{1.86}$$

and

$$(m =)\frac{\lambda}{c}\int_0^\infty ye^{Ry}(1 - B(y))dy < \infty, \quad (m =)\frac{\lambda}{c}\int_0^\infty ye^{Ry}(1 - B(y))dy = 1, \tag{1.87}$$

then, the following approximation formula is valid:

$$\Psi(u) \approx Ce^{-\lambda u} \tag{1.88}$$

where the constant C has as value:

$$C = \left(1 - \frac{\lambda\beta}{c}\right)(Rcm)^{-1}. \tag{1.89}$$

Proof It can be shown that under our assumptions, the function $[0,\infty) \mapsto \mathbb{R}^+$ **:**

$$y \mapsto e^{Ry}(1 - B^*(y)) \tag{1.90}$$

is directly Riemann integrable (see Çinlar (1975b) for the definition))
We can thus use the key renewal theorem (see **Proposition 4.2**, Chapter 2) to obtain from the integral equation of renewal type (1.82):

$$\lim_{t\to\infty}\hat{\Psi}(t) = \frac{1}{m}\int_0^\infty G(x)dx \tag{1.91}$$

with

$$G(x) = \frac{\lambda\beta}{c} e^{Rx}(1 - B^*(x)) \tag{1.92}$$

and

$$m = \frac{\lambda\beta}{c}\int_0^\infty e^{Ry}\,dB^*(y). \tag{1.93}$$

By integrating both members of relation (1.92), we get with an integration by parts for the second member:

$$\begin{aligned}\int_0^\infty G(x)dx &= \frac{\lambda\beta}{c}\int_0^\infty (1 - B^*(x))d\frac{e^{Rx}}{R} \\ &= \frac{\lambda\beta}{cR}(1 - B^*(x))e^{Rx}\Big|_0^\infty + \frac{\lambda\beta}{cR}\int_0^\infty e^{Rx}\,dB^*(x).\end{aligned} \tag{1.94}$$

By the direct Riemann integrability of function (1.90), the bracket has value zero at $+\infty$. Moreover, using assumption (1.86) to deal with the second term of the second member of this last equality, we get:

$$\begin{aligned}\int_0^\infty G(x)dx &= -\frac{\lambda\beta}{cR} + \frac{\lambda\beta}{cR}\frac{c}{\lambda\beta}, \\ &= \frac{1}{R}\left(1 - \frac{\lambda\beta}{c}\right).\end{aligned} \tag{1.95}$$

Using relations (1.80), (1.91) and (1.95) finally gives:

$$\begin{aligned}\lim_{u\to\infty}\Psi(u)e^{Ru} &= \frac{1}{mR}\left(1 - \frac{\lambda\beta}{c}\right), \\ &= C,\end{aligned} \tag{1.96}$$

and so the theorem is proved. □

Before giving the next result, we must first write the basic assumption (1.86) of **Proposition 1.1** under another form. To do this transformation, let us begin to express the integral

$$\int_0^\infty e^{Ry}\,dB^*(y) = \frac{1}{\beta}\int_0^\infty e^{Ry}(1 - B(y))dy; \tag{1.97}$$

using an integration by parts for the second member we obtain:

$$\int_0^\infty e^{Ry}\,dB^*(y) = -\frac{1}{\beta R} + \frac{1}{\beta R}\int_0^\infty e^{Ry}\,dB(y). \tag{1.98}$$

Thus, assumption (1.86) becomes:

$$\frac{c}{\lambda\beta} = -\frac{1}{\beta R} + \frac{1}{\beta R}\int_0^\infty e^{Ry}\,dB(y) \tag{1.99}$$

or equivalently:

$$\lambda + Rc = \lambda\int_0^\infty e^{Ry}\,dB(y). \tag{1.100}$$

This new form of the R-equation (1.100) is called the *Cramer-Lundberg equation*.

This equation clearly shows that the existence of a finite positive value of R implies the existence of the generating function of the d. f. B, at least on $[0,R]$, and consequently that this d.f. has moments of every order.
The Cramer-Lundberg equation has a simple geometric interpretation as the value of R is given by the strictly positive value of the intersection point of the curve representing the function $R \mapsto \lambda\int_0^\infty ye^{Ry}dB(y)$ and the straight line d whose equation is given by the first member of equation (1.100).
It follows that the slope of the tangent t at the curve C at the origin has the value $\lambda\beta$. From relation (1.85), this value is strictly less than the slope c of d. Moreover, it is easily seen that the function defined by the second member of the Cramer-Lundberg equation is a strictly increasing convex function. This implies that this equation has only one strictly positive solution R.
It can be shown that the value R is strictly less than 1 (see for example Gerber (1979)).
The next result gives an interesting upper bound of the ruin probability.

Corollary 1.1 *Under the assumptions of Proposition 1.1, the following inequality is true for all positive u:*

$$\Psi(u) \le e^{-Ru}. \tag{1.101}$$

Proof Using the r.v. defined by:

$$\begin{aligned} \widehat{Z}_n &= -Z_n (= cX_n - Y_n), \\ \widehat{S}_n &= \sum_{k=0}^{n} \widehat{Z}_n, n = 0,1,..., \\ \widehat{S}_0 &= 0, \end{aligned} \tag{1.102}$$

we can write for, $\Psi_n(u)$, the probability of being ruined by one of the first n claims (n=1,2,...):

$$\Psi_n(u) = P(\inf\left\{\widehat{S}_0,...,\widehat{S}_n\right\} \le -u). \tag{1.103}$$

Of course, since $\widehat{S}_0 = 0$ we can extend the definition of $\Psi_n(u)$ for negative values of u so that:

$$\Psi_n(u) = 1, u < 0. \tag{1.104}$$

Since

$$\Psi(u) = \lim_{n\to\infty} \Psi_n(u), \tag{1.105}$$

and since the ruin event can only occur upon the arrival of a claim, it suffices to show that for all real value of u:

$$\Psi_n(u) \le e^{-Ru}, n = 0,1,.... \tag{1.106}$$

For n=0, the result is obvious since:

$$\Psi_0(u) = \begin{cases} 0, u \geq 0, \\ 1, u \leq 0. \end{cases} \tag{1.107}$$

By induction, we get successively:

$$\begin{aligned} \Psi_{n+1}(u) &= \int_0^\infty \lambda e^{-\lambda t} dt \int_0^\infty \Psi_n(u + ct - y) dB(y) dt \\ &\leq \int_0^\infty \lambda e^{-\lambda t} dt \int_0^\infty e^{-R(u+ct-y)} dB(y) dt \\ &\leq e^{-Ru} \int_0^\infty \lambda e^{-(\lambda + Rc)t} dt \int_0^\infty e^{-Ry} dB(y) \\ &\leq e^{-Ru} \frac{\lambda}{\lambda + Rc} \int_0^\infty e^{-Ry} dB(y). \end{aligned} \tag{1.108}$$

It suffices now to use the Cramer-Lundberg equation (1.100) to obtain the desired result (1.101). □

This last corollary shows the fundamental importance of inequality (1.101). The knowledge of R gives to the insurer the possibility to adopt an informed, careful attitude and it is well known that carefulness is one of the pillars of insurance.
The price to pay is just the resolution of the Cramer-Lundberg equation but that is not a problem if we use the numerical Newton method.
However, the next two results show how to avoid this resolution in order to get other upper bounds of the ruin probability.

Corollary 1.2
(i) *Under the assumptions of* **Proposition 1.1**, *and if moreover, the variance* σ^2 *related to the d.f. B, is supposed to be finite, then:*

$$R < \frac{2(c - \lambda\beta)}{\lambda(\beta^2 + \sigma^2)}. \tag{1.109}$$

(ii) *If moreover the claim amount is, a.s., a bounded r.v. with M as upperbound, then:*

$$\frac{1}{M} \ln \frac{c}{\lambda\beta} < R. \tag{1.110}$$

Proof
(i) This result is easily obtained by replacing e^{Ry} in the equation (1.100) by its development into a second-order MacLaurin series.
(ii) The function $y \mapsto e^{Ry}$ being convex on $[0, \infty)$, we get from the convexity inequality that:

$$e^{Rx} \leq \frac{x}{M} e^{RM} + \left(1 - \frac{x}{M}\right), \ 0 \leq x \leq M. \quad e^{Rx} \leq \frac{x}{M} \int_0^M x. \tag{1.111}$$

Coming back to the Cramer Lundberg equation (1.100), we obtain:

$$\lambda + Rc \le \frac{\lambda}{M}\int_0^M xe^{RM}dB(x) + \lambda - \frac{\lambda}{M}\int_0^M xdB(x), \tag{1.112}$$

so that:

$$\lambda + Rc \le \frac{\lambda\beta}{M}e^{RM} + \lambda - \frac{\lambda\beta}{M}. \tag{1.113}$$

It follows that:

$$\frac{c}{\lambda\beta} \le \frac{e^{RM} - 1}{RM}, \tag{1.114}$$

and consequently:

$$\frac{c}{\lambda\beta} \le e^{RM}, \tag{1.115}$$

a result equivalent to the inequality (1.110) to be proved. □

So, under the assumption that there exists a constant M such that $B(M)=1$, then we have:

$$\frac{1}{M}\ln\frac{c}{\lambda\beta} < R < \frac{2(c-\lambda\beta)}{\lambda(\beta^2+\sigma^2)}. \tag{1.116}$$

In particular, by **Corollary 1.1**, we have:

$$\Psi(t) < e^{-\frac{t}{M}\ln\frac{c}{\lambda\beta}}. \tag{1.117}$$

This last inequality gives a useful upper bound of the ruin probability without having to solve the Cramer-Lundberg equation. of course, as M tends toward $+\infty$, we get the trivial bound 1.

Example 1.3 Let us suppose that the claim amount distribution is normal $(\beta,\sigma^2),\beta > 0$ and with μ large enough so that the truncated normal is well approximated by the entire normal curve.

From result (5.13) of Chapter 1, the Cramer-Lundberg equation (1.100) becomes:

$$\lambda + Rc = \lambda e^{\beta R + \frac{\sigma^2 R^2}{2}}. \tag{1.118}$$

Using a Taylor development of order 1, we get:

$$R \approx 2\frac{c-\lambda\beta}{\sigma^2}. \tag{1.119}$$

Example 1.4 *(The adjustment coefficient for the P/P risk model)*

In this case, we know that:

$$\varphi_B(s) = \frac{1}{1-s\beta}, 0 \le s < \frac{1}{\beta}. \tag{1.120}$$

For this special case, the Cramer-Lundberg (1.100) equation takes the form:

$$\lambda + Rc = \frac{\lambda}{1 - R\beta}. \tag{1.121}$$

Therefore, the value of the adjustment coefficient is

$$R = \frac{\eta}{1+\eta}\beta \tag{1.122}$$

where we know that:

$$\eta = 1 - \frac{\lambda\beta}{c}. \tag{1.123}$$

This last result gives the following Cramer-Lundberg inequality (1.100) for the *P/P* model:

$$\psi(u) \le \exp\left(-\frac{\eta}{1+\eta}\frac{u}{\beta}\right). \tag{1.124}$$

Comparing with the exact value of ψ given by relation (1.68), we see that this majoration gives the overestimation:

$$\frac{\eta}{1+\eta}\exp\left(-\frac{\eta}{1+\eta}\frac{u}{\beta}\right). \tag{1.125}$$

Remark 1.1

(i) *Economic interpretation of the adjustment coefficient*

Using the exponential utility principle to cover the total outcome U(t) on the time interval $[0,t]$, we can write:

$$E(e^{aU(t)}) = e^{aP(t)} \tag{1.126}$$

where $P(t)$ represents the mean global premium for the insurance company.

To find the value of the parameter a satisfying this last relation, let us first recall that, from relation (1.19), we have:

$$P(U(t) \le y) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}B^{(n)}(y). \tag{1.127}$$

It follows that the generating function of the r.v. $U(t)$ is given by:

$$\begin{aligned} E(e^{sU(t)}) &= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\rho_B^n(s) \\ &= e^{-\lambda t(1-\varphi_B(s))}. \end{aligned} \tag{1.128}$$

where the function ρ_B represents the generating function associated with the r.v. Y_n.

Consequently, the equality (1.126) is equivalent to:

$$-\lambda t(1-\rho_B(a)) = aP(t). \tag{1.129}$$

We want to have a risk model for which the ratio $P(t)/t$ is independent of t and has as value c. Therefore, we must have from relation (1.129):

$$-\lambda t(1-\rho_B(a)) = act, \tag{1.130}$$

which is the Cramer Lundberg equation for the unknown parameter a, so that a=R.
This result shows that if the choice is a linear income $P(t) = ct$, then a, the relative marginal utility of the policyholder, is the constant R.
(ii) *The adjustment coefficient and martingale theory*
It is also possible to show that the adjustment coefficient R is the only value for a such that the process $(e^{-aU(t)}, t \geq 0)$ is a martingale with respect to the filtration

$$\begin{aligned}&(\Im_t, t \geq 0),\\&\Im_t = \sigma(U(s), s \leq t)\end{aligned} \tag{1.131}$$

and so, in particular:

$$E(e^{-RU(t)}) = e^{-Ru}. \tag{1.132}$$

2 DIFFUSION MODELS FOR RISK THEORY AND RUIN PROBABILITY

There exist other risk models than those of type *G/G*. Models using semi-Markov processes will be fully developed later.
In this section, we will briefly present two models using simple diffusion processes (see Chapter 1).

2.1 The Simple Diffusion Risk Model

In this model (Cox and Miller (1965) and Gerber(1979)), we will immediately model the risk reserve or the surplus process with a particular continuous time stochastic process.
This means that the α process satisfies the very simple stochastic differential equation:

$$\begin{aligned}&d\alpha = \mu dt + \sigma dW(t),\\&\alpha(0) = u.\end{aligned} \tag{2.1}$$

The process $W = (W(t), t \geq 0)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \Im, P)$ and of course, we suppose that:

$$\mu > 0, \sigma > 0. \tag{2.2}$$

As in continuous time stochastic finance (see Merton(1999)), the first parameter is called the *trend* and the second one the *volatility*.
This model gives a very simple expression for the α process:

$$\alpha(t) = \mu t + \sigma W(t), t \geq 0. \tag{2.3}$$

With such a simple model, it is possible to compute the exact value of the ruin probability (see Cox and Miller (1965)) on a finite time horizon [0,t], that is

$$\psi(u,t) = P(T < t \mid \alpha(0) = u), \tag{2.4}$$

with T defined by relation (1.22) and $\psi(u,t)$ by the following expression:

$$\psi(u,t)=1-\bar{\phi}\left(\frac{u+\mu t}{\sigma\sqrt{t}}\right)+e^{-\frac{2\mu}{\sigma^2}u}\bar{\phi}\left(\frac{-u+\mu t}{\sigma\sqrt{t}}\right), \tag{2.5}$$

where here, to avoid confusion with the notation for the non-ruin probability, $\bar{\phi}$ represents the d.f. of a reduced normal r.v.

Let us point out that, letting $t\to\infty$, we get the following asymptotic result:

$$\psi(u)=\lim_{t\to\infty}\psi(u,t)=\begin{cases} e^{-\frac{2\mu}{\sigma^2}u}, \mu>0, \\ 1, \mu<0. \end{cases} \tag{2.6}$$

Remark 2.1

a) From the result (2.5), we deduce that, for all positive u:

$$\lim_{t\to 0}\psi(u,t)=0\,. \tag{2.7}$$

This is a simple consequence of the a.s. global continuity property of the trajectories of the α process. That was not the case for the *G/G* risk model!

b) We also have:

$$\psi(0,t)=1, t>0 \tag{2.8}$$

and so:

$$\psi(0)=1\,, \tag{2.9}$$

contrary to the result (1.116) for the Cramer-Lundberg or G/G risk model.

c) The asymptotic result (2.6) gives interesting information concerning the strategic point of view of the insurance company.

Indeed, in this formula (2.6), the basic parameter is $2\mu/\sigma^2$.

It gives a good measure of the two models of action available to the manager of the company: increase or decrease the premiums, i.e. act on the trend μ, or increase or decrease the risk by the mix of portfolio selection, i.e. act on the volatility σ.

2.2 The ALM-like Risk Model (Janssen(1991),(1993))

In finance, it is usual to model the evaluation of the assets and the liabilities of a bank or of an insurance company with the use of stochastic processes for both parts of the balance sheet. This leads to useful models used in the theory and practice of *asset liability management* (in short ALM (Janssen (1991), (1993)).

We will now briefly present this type of model for an insurance company.

Let us represent by

$$A=(A(t),t\geq 0), B=(B(t),t\geq 0) \tag{2.10}$$

successively the stochastic processes of the *asset* and of the *liability* under the assumption that they satisfy the very simple stochastic differential system

$$\begin{aligned} &dA(t) = \mu_A dt + \sigma_A dW_A(t), \\ &dB(t) = \mu_B dt + \sigma_B dW_B(t), \\ &A(0) = u, B(0) = 0 \end{aligned} \tag{2.11}$$

and where:

(i) $\mu_A, \mu_B, \sigma_A, \sigma_B, u$ are strictly positive,

(ii) $W_A = (W_A(t), t \geq 0), W_B = (W_B(t), t \geq 0)$ are two independent standard Brownian motions.

Clearly, the stochastic differential problem (2.11), has the following solution:

$$\begin{aligned} &A(t) = u + \mu_A t + \sigma_A W_A(t), \\ &B(t) = \mu_B t + \sigma_B W_B(t). \end{aligned} \tag{2.12}$$

As above, we also have:

$$\begin{aligned} &\psi(u,t) = P(T \leq t \,|\, A(0) = u), \\ &\psi(u) = P(T \leq \infty \,|\, A(0) = u) = \lim_{t \to \infty} \psi(u,t). \end{aligned} \tag{2.13}$$

From (2.12), we can write that:

$$A(t) - B(t) = u + \mu_A t + \sigma_A W_A(t) - \mu_B t - \sigma_B W_B(t). \tag{2.14}$$

The independence assumption between the two Brownian processes implies that the process

$$(AW_A(t) - BW_B(t), t \geq 0) \tag{2.15}$$

is probabilistically equivalent to the process

$$\left(\sqrt{\sigma_A^2 + \sigma_B^2}\, W(t), t \geq 0\right) \tag{2.16}$$

where the process W is a standard Brownian motion.

Let us now introduce two new parameters defined as:

$$\mu = \mu_A - \mu_B, \sigma = \sqrt{\sigma_A^2 + \sigma_B^2}, \tag{2.17}$$

so that, using relation (2.14), we get

$$A(t) - B(t) = \mu t + \sigma W(t). \tag{2.18}$$

Thus, we see that, with the change of parameters (2.17), the process $(A - B)$ is modelled exactly like the simple diffusion risk model given by relation (2.3). Consequently, all the results of subsection 2.1 are valid for the ALM-like model, particularly the results (2.5), (2.6) giving here as results:

$$\psi(u,t) = 1 - \bar{\phi}\left(\frac{u + (\mu_A - \mu_B)t}{\sqrt{(\sigma_A^2 + \sigma_B^2)t}}\right) + e^{-\frac{2\mu}{(\sigma_A^2+\sigma_B^2)}u}\,\bar{\phi}\left(\frac{-u + (\mu_A - \mu_B)t}{\sqrt{(\sigma_A^2 + \sigma_B^2)t}}\right), \tag{2.19}$$

$$\psi(u) = \lim_{t \to \infty} \psi(u,t) = \begin{cases} e^{-2\frac{\mu_A - \mu_B}{\sigma_A^2 + \sigma_B^2}u} & , \mu_A > \mu_B, \\ 1 & , \mu_A < \mu_B. \end{cases} \tag{2.20}$$

Remark 2.2
a) In addition to **Remark 2.1**, let us say that, with the ALM-like risk model, the manager has more flexibility to measure the influence of strategic changes. This is due to the fact that there are four parameters with two of them, μ_A, σ_A only for the asset part and the last two, μ_B, σ_B , only for the liability part.
b) For the ALM-like risk model, the basic parameter R becomes:

$$R = 2\frac{\mu_A - \mu_B}{\sigma_A^2 + \sigma_B^2}. \tag{2.21}$$

It gives the possibility to correct a bad change, for example in the asset part with an action on the liability part or vice-versa, and to introduce *hedging strategies* for insurance companies (see Janssen, Bergendhäl (1999)).

2.3 Comparison Of ALM-Like And Cramer-Lundberg Risk Models

It is interesting to see how "to adjust" an ALM risk model to a *P/G* or Cramer-Lundberg model.
To do this, let us begin to consider a *P/G* model with as basic parameters λ, β, η .
From result (1.127), we know that the d.f. of $U(t)$, the total claim amounts up to time t, is given by

$$P(U(t) \le y) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} B^{(n)}(y). \tag{2.22}$$

From this expression, it results that:

$$E(U(t)) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} n\beta = \lambda\beta t. \tag{2.23}$$

Using the second Wald's identity (Chapter 1, relation (6.59)), we also have

$$\begin{aligned} \operatorname{var}(U(t)) &= \operatorname{var}(Y_1)E(N(t)) + (E(Y_1))^2 \operatorname{var}(N(t)) \\ &= \lambda\sigma^2 t + \lambda\beta^2 t \\ &= \alpha_2 \lambda t, \end{aligned} \tag{2.24}$$

σ^2 and α_2 being respectively the variance and the centred moment of order 2 related to the d.f. B.
Let us note respectively by $\alpha_{CL}, \alpha_{ALM}$ the risk reserve processes related to the Cramer-Lundberg model and the ALM risk model. From relation (2.23), it follows that:

$$\begin{aligned} E(\alpha_{CL}(t)) &= E(u + ct - U_{CL}(t)) \\ &= u + ct - \lambda\beta t. \end{aligned} \tag{2.25}$$

For the variance, we obtain:

$$\begin{aligned}\text{var}(\alpha_{CL}(t)) &= \text{var}(U_{CL}(t)) \\ &= \lambda\alpha_2 t.\end{aligned} \tag{2.26}$$

For the ALM risk model, we have:

$$\alpha_{ALM}(t) = u + A(t) - B(t), \tag{2.27}$$

and consequently:

$$\begin{aligned}E(\alpha_{ALM}(t)) &= u + (\mu_A - \mu_B)t, \\ \text{var}(\alpha_{ALM}(t)) &= (\sigma_A^2 + \sigma_B^2)t.\end{aligned} \tag{2.28}$$

Remembering that $c = \lambda\beta(1+\eta)$, the method of moments for the adjustment of these two models gives, with relations (2.25), (2.26) and (2.28), the following conditions:

$$\begin{aligned}\lambda\beta\eta &= \mu_A - \mu_B, \\ \lambda\alpha_2 &= \sigma_A^2 + \sigma_B^2.\end{aligned} \tag{2.29}$$

The key parameter (2.21) is given by:

$$R = 2\frac{\mu_A - \mu_B}{\sigma_A^2 + \sigma_B^2} = 2\frac{\beta\eta}{\alpha_2}. \tag{2.30}$$

From result (2.20), we can now propose an approximation value for the ruin probability $\psi_{CL}(u)$ which is given by:

$$\psi_{CL}(u) \simeq \exp\left(-2\frac{\beta\eta}{\alpha_2}u\right). \tag{2.31}$$

Of course, this approximation will be reliable only if the ALM risk model fits well the *G/P* considered model.

Substituting the value of $\eta = \dfrac{c - \lambda\beta}{\lambda\beta}$ in this last result, we get:

$$\psi_{CL}(u) \simeq \exp\left(-2\frac{c-\lambda\beta}{\lambda\alpha_2}u\right). \tag{2.32}$$

We see that the approximation (2.31) is equivalent to the approximation using inequality (1.109), showing so that the approximation with the ALM-like model gives a lower bound of ψ.

2.4 The Second ALM-Like Risk model

In the first ALM-like model, it is possible to have negative values for the asset and liability values. To avoid this eventuality, we will start with a new model called the second ALM-like Risk model or model ALM II based on the following stochastic differential equations under the same assumptions as for the first ALM-like model, called now ALM I but with dependence between the considered Brownian motions.

$$dA = A\mu_A dt + A\sigma_A^{'} dW + A\beta_A dZ_A^{'}, \tag{2.33}$$
$$dB = B\mu_B dt + B\sigma_B^{'} dW + B\beta_B dZ_B^{'}, \tag{2.34}$$

where

(i) W=($W(t)$, $t \geq 0$) is a standard Brownian motion,

(ii) $Z' = (Z_A^{'}(t), Z_B^{'}(t), t \geq 0)$ is a two-dimensional standard Brownian motion with as covariance matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, |\rho| \leq 1, \tag{2.35}$$

(iii) the processes W and Z' are independent,

(iv) $A(0) = A_0, B(0) = B_0, A_0 > B_0,$ (2.36)

(v) μ_A, μ_B, σ_A, σ_B are non-negative parameters.

Without loss of generality, the model (2.33), (2.34) can be replaced by the following one:

$$\begin{aligned} dA &= A\mu_A dt + A\sigma_A dZ_A, \\ dB &= B\mu_B dt + A\sigma_B dZ_B, \end{aligned} \tag{2.37}$$

where

$$\begin{aligned} \sigma_A^2 &= \sigma_A^{'2} + \beta_A^2, \\ \sigma_B^2 &= \sigma_B^{'2} + \beta_B^2, \end{aligned} \tag{2.38}$$

Z_A, Z_B being two standard Brownian motions with **M** as correlation coefficient.

With the result (4.20) of Chapter 5, we get:

$$\frac{A(t)}{B(t)} = \frac{A_0}{B_0} \exp(\mu t + \sigma \tilde{W}(t)) \tag{2.39}$$

with

$$\frac{A(t)}{B(t)} = \frac{A_0}{B_0} \exp(\mu t + \sigma \tilde{W}(t)), \tag{2.40}$$

$$\mu = \mu_A - \mu_B - \frac{1}{2}(\sigma_A^2 - \sigma_B^2), \tag{2.41}$$

$$\sigma^2 = \sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B. \tag{2.42}$$

Let us remark that, theoretically, the risk component disappears when $\sigma_A = \sigma_B, \rho = 1$. Of course, this case never occurs in practice but it means that the management of the company must tend toward this result if they want to minimize the risk.

As before we can study the lifetime of the company T, here defined as the first value of t such that $A(t) < B(t)$, or equivalently such that the process

$$\left(\ln \frac{A(t)}{B(t)}, t \geq 0 \right) \tag{2.43}$$

hits $(-\infty,0)$.
We can thus write:

$$T = \inf\{t - \mu t - \sigma\tilde{W}(t) > a\},$$
$$a = \ln\frac{A_0}{B_0}. \tag{2.44}$$

Using the result of section 2.1, we get:

$$P(T<\infty) = \begin{cases} 1, \mu \le 0, \\ e^{-2\frac{\mu}{\sigma^2}a}, \mu > 0. \end{cases} \tag{2.45}$$

We can also get results for the following probability:

$$P(x,t) = P(T > t, M(t) \le x), x < a,$$
$$M(t) = -\mu t - \sigma\tilde{W}(t). \tag{2.46}$$

We can also write:

$$P(x,t) = P(T > t, e^{a-x} < M(t) \le x), x < \frac{A(t)}{B(t)}), x < a. \tag{2.47}$$

Cox and Miller (1965) proved that

$$P(x,t) = \int_{-\infty}^{x} p(y,t)dy, \tag{2.48}$$

$$p(y,t) = \frac{1}{\sigma\sqrt{2\pi t}}\exp\left[-\frac{(y+\mu t)^2}{2\sigma^2 t} - \frac{2\mu a}{\sigma^2}\frac{(y-2a+\mu t)^2}{2\sigma^2 t}\right] \tag{2.49}$$

and by integration:

$$P(x,t) = \bar{\phi}\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right) - e^{-2\frac{\mu}{\sigma^2}a}\bar{\phi}\left(\frac{x-2a+\mu t}{\sigma\sqrt{t}}\right). \tag{2.50}$$

Here, the non-ruin probability $\phi(a,t)$ on $[0,t]$ is given by:

$$\phi(a,t) = P(a,t) \tag{2.51}$$

and so

$$\phi(a,t) = \bar{\phi}\left(\frac{a+\mu t}{\sigma\sqrt{t}}\right) - e^{-2\frac{\mu}{\sigma^2}a}\bar{\phi}\left(\frac{-a+\mu t}{\sigma\sqrt{t}}\right) \tag{2.52}$$

for which we get result (2.44), letting $t \to \infty$.
When the ruin is certain on an infinite horizon, that is iff $\mu \le 0$, T has the following density:

$$f_T(t) = \frac{a}{\sigma\sqrt{2\pi t^3}}e^{-\frac{(a+\mu t)^2}{2\sigma^2 t}}, \tag{2.53}$$

called an inverse Gaussian distribution for which it can be proved that

$$E(T) = \frac{a}{|\mu|}, \operatorname{var}(T) = \frac{a\sigma^2}{|\mu^3|}. \tag{2.54}$$

Remark 2.3
(i) The more negative the correlation coefficient is, the riskier the situation is for the company.
(ii) As a function of the correlation coefficient, extreme possibilities are:
1) *"maximum"risk* $(\rho = -1)$

$$\mu + \frac{\sigma^2}{2} = (\mu_A + \mu_B) + (\sigma_A + \sigma_B)^2. \tag{2.55}$$

2) *"minimum" risk* (ρ = 1)

$$\mu + \frac{\sigma^2}{2} = (\mu_A - \mu_B) + (\sigma_A - \sigma_B)^2, \tag{2.56}$$

and if moreover the asset and liabilities volatilities are identical, we have for $\rho = -1$:

$$\mu + \frac{\sigma^2}{2} = (\mu_A - \mu_B) + 2\sigma_A^2 \tag{2.57}$$

and for $\rho = 1$

$$\mu + \frac{\sigma^2}{2} = \mu_A - \mu_B. \tag{2.58}$$

(iii) As the r.v. $A(t)A_0/B(t)B_0$ has a log-normal distribution $(\mu, \sigma^2 t)$, we have:

$$\begin{aligned} E\left(\frac{A(t)}{B(t)}\right) &= \frac{A_0}{B_0} e^{\left(\mu + \frac{\sigma^2}{2} t\right)}, \\ \operatorname{var}\left(\frac{A(t)}{B(t)}\right) &= \left(\frac{A_0}{B_0}\right)^2 e^{2\mu t} e^{\sigma^2 t} (e^{\sigma^2 t} - 1). \end{aligned} \tag{2.59}$$

These results confirm a well-known fact for investors: the larger the risk is, the larger the expectation of profit is too.
On the other hand, in all cases, the most dangerous situation happens when $\mu \le 0$, that is when

$$(\mu_A - \mu_B) \le \frac{1}{2}(\sigma_A^2 - \sigma_B^2). \tag{2.60}$$

In this case, the manager must absolutely diminish this excess of risk with an increase of $\mu_A - \mu_B$ and a lowering of the volatility of the assets together with an increase of the volatility of the liabilities.

3 SEMI- MARKOV RISK MODELS

In this paragraph, we will give a complete presentation of the so-called homogeneous semi-Markov risk model (in short SMRM) first introduced by Miller (1962) and fully developed by Janssen (1969b, 1970, 1977) and later many other authors.
We will also develop special cases of interest which bring more tractable results.

3.1 The Semi-Markov Risk Model (or SMRM)

As we already know from section 1 of this chapter, any risk model is based on three "basic" processes:

(i) the claim arrival process,
(ii) the claim amount process,
(iii) the premium income.

In general, the first two processes are stochastic processes and the last one deterministic.
These processes are defined on a complete probability space $\left(\Omega, \Im, P\right)$.

3.1.1 The general SMR Model

In the SMRM, the first idea was to introduce m possible *types of claims* belonging to the set

$$I = \{1,...,m\} \tag{3.1}$$

and later (see Janssen and Reinhard (1982)) this set was considered as an *environment parameter* and in both cases as having influences on the three basic processes given above.
Let $\left(X_n, n \geq 1\right), \left(Y_n, n \geq 1\right)$ represent respectively the sequence of interarrival times between two successive claims and the sequence of successive claim amounts. The process $\left(J_n, n \geq 1\right)$ will represent the successive type of claims or environment states.
The *basic assumption* to get an SMRM is that:

$$P(J_n = j, X_n \leq x, Y_n \leq y \,|\, (J_k, X_k, Y_k,), k = 1,...,n-1) = Q_{J_{n-1}j}(x, y) \tag{3.2}$$

with

$$J_0 = j_0, X_0 = Y_0 = 0, a.s. \tag{3.3}$$

This assumption means that the three-dimensional process $((J_n, X_n, Y_n), n \geq 0)$ is what is called a two-dimensional (*J*-*X*)-process of *kernel* **Q**, having the following properties:

(i) all the elements Q_{ij} of **Q** are mass functions in two dimensions, null for *x* or *y* negative,
(ii) the following limits exist:

$$\lim_{x\to\infty, y\to\infty} Q_{ij}(x,y) = p_{ij}, i, j \in I,$$
$$\sum_{j=1}^{m} p_{ij} = 1, i \in I. \tag{3.4}$$

Every such matrix **Q** is called a *two-dimensional semi-Markov kernel* and the corresponding (*J-X-Y*) process a *two-dimensional J-X process* or a *two-dimensional semi-Markov chain.*
From a straightforward extension of the basic results of Chapter 3, section 2, we get the following results:
(i) the process of successive claims $\left(J_n, n \geq 0\right)$ is a homogeneous Markov chain with *I* as state space and with $\mathbf{P} = \left[p_{ij}\right]$ as transition matrix,
(ii) the processes $((J_n, X_n), n \geq 0), ((J_n, Y_n), n \geq 0)$ are two SMP of kernels ${}^{A}\mathbf{Q}$, ${}^{B}\mathbf{Q}$ where for all *i* and *j* of *I*:

$${}^{A}Q_{ij}(x) = Q_{ij}(x, +\infty),\ {}^{B}Q_{ij}(y) = Q_{ij}(+\infty, y), \tag{3.5}$$

(iii) given the r.v. $J_n, n \geq 0$ the two-dimensional r.v. $(X_n, Y_n), n \geq 1$ are conditionally independent and we have:

$$F_{ij}(x,y) = P(X_n \leq x, Y_n \leq y \,|\, J_0, ..., J_{n-2}, J_{n-1} = i, J_n = j)$$
$$= \begin{cases} Q_{ij}(x,y)/p_{ij}, p_{ij} > 0, \\ U_1(x)U_1(y), p_{ij} = 0. \end{cases} \tag{3.6}$$

(iv) From this last property, we see that given the r.v. $J_n, n \geq 0$, the r.v. $(X_n, n \geq 1)$ are conditionally dependent and similarly for the r.v. $(Y_n, n \geq 1)$ and moreover:

$${}^{A}F_{ij}(x) = P(X_n \leq x, \,|\, J_0, ..., J_{n-2}, J_{n-1} = i, J_n = j) = F_{ij}(x, +\infty),$$
$${}^{B}F_{ij}(y) = P(Y_n \leq y, \,|\, J_0, ..., J_{n-2}, J_{n-1} = i, J_n = j) = F_{ij}(+\infty, y). \tag{3.7}$$

Suppressing the conditioning relative to J_n, we get

$$H_i(x,y) = P(X_n \leq x, Y_n \leq x \,|\, J_0, ..., J_{n-2}, J_{n-1} = i) = \sum_j p_{ij} F_{ij}(x,y),$$
$${}^{A}H_i(x) = P(X_n \leq x, \,|\, J_0, ..., J_{n-2}, J_{n-1} = i) = H_i(x, +\infty), \tag{3.8}$$
$${}^{B}H_i(y) = P(Y_n \leq y, \,|\, J_0, ..., J_{n-2}, J_{n-1} = i) = H_i(+\infty, y).$$

Now, we can introduce the means associated with the different conditional d.f. defined above and we adopt the following notation:

$$a_{ij} = \int_0^\infty x d\,{}^A F_{ij}(x), b_{ij} = \int_0^\infty y d\,{}^B F_{ij}(y),$$
$${}^A\eta_i = \int_0^\infty x d\,{}^A H_i(x)\left(= \sum_{j=1}^m p_{ij} a_{ij} \right),\ {}^B\eta_j = \int_0^\infty y d\,{}^B H_i(y)\left(= \sum_{j=1}^m p_{ij} b_{ij} \right). \tag{3.9}$$

Now, we can first introduce the process

$$(T_n, n \geq 1), \quad T_0 = 0 \tag{3.10}$$

defined as

$$T_n = \sum_{k=1}^n X_k, n \geq 1, \tag{3.11}$$

representing the successive *claim time arrivals* and secondly the process

$$(U_n, n \geq 1), \quad U_0 = 0 \tag{3.12}$$

defined as

$$U_n = \sum_{k=1}^n Y_k, n \geq 1, \tag{3.13}$$

representing the successive total *claim amount just after the arrivals of the successive claims.*

For the joint distribution of the process $(J_n, T_n, U_n, n \geq 0)$, we get:

$$\begin{aligned} &P(J_n = j, T_n \leq t, U_n \leq y \mid J_0 = i) = Q_{ij}^{(n)}(x, y), \\ &Q_{ij}^{(0)}(x, y) = \delta_{ij} U_0(x) U_0(y), \\ &Q_{ij}^{(1)}(x, y) = Q_{ij}(x, y), \\ &Q_{ij}^{(n)}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \sum_{k=1}^{n} Q_{ij}^{(n-1)}(x - x', y - y') Q(dx', dy'), n > 1. \end{aligned} \tag{3.14}$$

Of course, for processes $((J_n, T_n), n \geq 0), ((J_n, U_n), n \geq 0)$, both MRP, we have:

$$\begin{aligned} &P(J_n = j, T_n \leq t \mid J_0 = i) = {}^A Q_{ij}^{(n)}(t), \\ &P(J_n = j, U_n \leq y \mid J_0 = i) = {}^B Q_{ij}^{(n)}(y). \end{aligned} \tag{3.15}$$

Remark 3.1 By analogy with the basic definitions for SMP given in Chapter 3, section 2 (**Definition 2.1**), the three-dimensional process $((J_n, T_n, U_n), n \geq 0)$ is called a *two- dimensional MRP of kernel* **Q**.

If **Q** is an extended SM matrix in two dimensions, this process is called a *two-dimensional MRW* or *extended SMC.*

We finish this section with the following definition.

Definition 3.1 *The sequences* $(X_n, n \geq 1), (Y_n, n \geq 1)$ *are conditionally independent given the sequence* $(J_n, n \geq 0)$ iff

$$F_{ij}(x,y) = {}^{A}F_{ij}(x)\,{}^{B}F_{ij}(y), \forall x,y \in R, \forall i,j \in I. \tag{3.16}$$

From the results given above in this subsection, we also have:

$$\begin{aligned}(3.16) &\Leftrightarrow Q_{ij}(x,y) = {}^{A}F_{ij}(x)\,{}^{B}Q_{ij}(y) \Leftrightarrow Q_{ij}(x,y) = {}^{A}Q_{ij}(x)\,{}^{B}F_{ij}(y)\\ &\Leftrightarrow Q_{ij}(x,y) = p_{ij}\,{}^{A}F_{ij}(x)\,{}^{B}F_{ij}(y).\end{aligned} \tag{3.17}$$

This assumption seems quite reasonable in risk theory and moreover, it may be useful to consider the following particular case:

$$\begin{aligned}{}^{A}F_{ij}(x) &= {}^{A}F_{j}(x), i,j \in I, x \geq 0,\\ {}^{B}F_{ij}(y) &= {}^{B}F_{j}(y), i,j \in I, y \geq 0.\end{aligned} \tag{3.18}$$

The first type of condition (3.18) means that the d.f. of the interarrival time between two consecutive claims uniquely depends upon the type of the future claim and the second one that the d.f. of a claim amount uniquely depends upon the type of this claim and not on the type of the preceding one.

3.1.2 The Counting Claim Process

Still using notation and concepts of Chapter 3, section 5, let us introduce the $m+1$ counting processes associated with the SMP of claim arrivals of kernel ${}^{A}\mathbf{Q}$:

$$\left({}^{A}N_j(t), t \geq 0\right), j = 1,...,m, \left({}^{A}N(t), t \geq 0\right), \tag{3.19}$$

so that here, ${}^{A}N_j(t)$ represents the total number of claims of type i occurring on $(0,t]$ and ${}^{A}N(t)$ represents the total number of claims occurring on $(0,t]$.

From now on, we will suppose that the MRP of kernel ${}^{A}\mathbf{Q}$ is ergodic with $\pi = (\pi_1,...,\pi_m)$ as a unique stationary distribution related to $\mathbf{P}$.

For $R_{ij}(t) = E({}^{A}N_j(t) \mid J_0 = i)$, we know from relations (6.15), (9.1) and (9.4) of Chapter 3 that

$$\begin{aligned}R_{ij}(t) &= \sum_{n=0}^{\infty} {}^{A}Q_{ij}^{(n)}(t), i,j \in I,\\ \lim_{t\to\infty} \frac{R_{ij}(t)}{t} &= \frac{1}{{}^{A}\mu_{jj}}, i,j \in I,\\ {}^{A}\mu_{jj} &= \frac{1}{\pi_j}\sum_k \pi_k\,{}^{A}\eta_k, j \in I.\end{aligned} \tag{3.20}$$

From now on, we will drop the index A for the counting variables related to the claim arrivals.

For the joint distribution of $\left(N(t), J_{N(t)}, T_{N(t)}\right)$, using the semi-Markov property, we successively get:

$$\begin{aligned}
&P\left(N(t)=n, J_{N(t)}=j, T_{N(t)} \le t-h \middle| J_0=i\right)\\
&=P\left(N(t)=n, J_n=j, T_n \le t-h \middle| J_0=i\right), 0 \le h \le t\\
&=P\left(T_n \le t < T_{n+1}, J_n=j, T_n \le t-h \middle| J_0=i\right) \qquad (3.21)\\
&=P\left(T_n \le t-h, T_{n+1} > t, J_n=j \middle| J_0=i\right)\\
&=\int_0^{t-h}(1-{}^AH_j(t-z))d\,{}^AQ_{ij}^{(n)}(z).
\end{aligned}$$

For h=0, we obtain:

$$P\left(N(t)=n, J_{N(t)}=j \middle| J_0=i\right)=\int_0^t(1-{}^AH_j(t-z))d\,{}^AQ_{ij}^{(n)}(z) \qquad (3.22)$$

and moreover, summing over j, we get:

$$\begin{aligned}
P\left(N(t)=n \middle| J_0=i\right)&=\sum_{j=1}^m\int_0^t(1-{}^AH_j(t-z))d\,{}^AQ_{ij}^{(n)}(z)\\
&=\sum_{j=1}^m {}^AQ_{ij}^{(n)}(t)-\sum_{j=1}^m\int_0^t\sum_{k=1}^m {}^AQ_{jk}(t-z)d\,{}^AQ_{ij}^{(n)}(z) \qquad (3.23)\\
&=\sum_{j=1}^m {}^AQ_{ij}^{(n)}(t)-\sum_{k=1}^m {}^AQ_{ij}^{(n+1)}(t).
\end{aligned}$$

Using the following notation:

$$\begin{aligned}
&{}^AP_{ij}(t,n)=P\left(N(t)=n, J_{N(t)}=j \middle| J_0=i\right),\\
&{}^AP_i(t,n)=P\left(N(t)=n, \middle| J_0=i\right)\left(=\sum_{j=1}^m {}^AP_{ij}(t,n)\right),
\end{aligned} \qquad (3.24)$$

we obtain from relations (3.23), the following recurrence formulas:

$$\begin{aligned}
&{}^AP_{ij}(t,n)=\sum_{k=1}^m {}^AP_{kj}(t,n-1)\bullet {}^AQ_{ik}(t),\\
&{}^AP_i(t,n)=\sum_{k=1}^m {}^AP_k(t,n-1)\bullet {}^AQ_{ik}(t),
\end{aligned} \qquad (3.25)$$

with of course:

$$\begin{aligned}
&P_{ij}(t,0)=\delta_{ij}(1-{}^AH_i(t)),\\
&P_i(t,0)=1-{}^AH_i(t).
\end{aligned} \qquad (3.26)$$

If we are only interested in one type of claim, say j, it suffices to consider the delayed renewal process characterized by $\left({}^AG_{ij,}\ {}^AG_{jj}\right)$. In this case we get

$$P(N_j(t)=n\,|\,J_0=i)=\begin{cases}1-{}^{A}G_{ij}(t),n=0,\\ {}^{A}G_{ij}\bullet({}^{A}G_{jj}^{(n-1)}-{}^{A}G_{jj}^{(n-1)})(t),n\geq 1.\end{cases} \tag{3.27}$$

For the means ${}^{A}H_{ij}(t)=E(N_j(t)\,|\,J_0=i)$, results (9.7) and (3.9) of Chapter 2 give:

$$\begin{aligned}H_{ij}(t)&=G_{ij}(t)+G_{ij}\bullet H_{jj}(t),i\neq j,\\ H_{jj}(t)&=\sum_{n=1}^{\infty}H_{jj}^{(n)}(t).\end{aligned} \tag{3.28}$$

Finally, **Proposition 7.2** of Chapter 2 is interesting here to get asymptotic normality:

$$N_j(t)\prec N\left(\frac{t}{\mu_{jj}},\frac{t\sigma_j^2}{\mu_{jj}^3}\right), \tag{3.29}$$

μ_{jj},σ_j^2 being respectively the mean and the variance of the r.v. ${}^{A}T_n(j\,|\,j)$ defined in Chapter 3, section 6, the latter supposed to be finite.

3.1.3 The Accumulated Claim Amount Process

This is the process $(U(t),t\geq 0)$ where:

$$U(t)=\sum_{n=1}^{N(t)}Y_n(=U_{N(t)}). \tag{3.30}$$

We already know that the marginal distribution of *U*(*t*), for fixed *t*, is important for insurance companies, as for example with a year as time unit, the value *U*(1) represents the total expenses of the company for paying the claims in this year *t*. Also, let us introduce the following marginal distributions:

$$M_{ij}(t,y)=P(U(t)\leq y,J_{N(t)}=j\,|\,J_0=i), \tag{3.31}$$

so that

$$\begin{aligned}M_{ij}(t,y)&=P(U(t)\leq y,J_{N(t)}=j\,|\,J_0=i),\\ M_{ij}(t,y)&=\sum_{n=0}^{\infty}\int_0^t(1-{}^{A}H_j(t-z))dQ_{ij}^{(n)}(z,y),\\ M_{ij}(t,y)&=\sum_{n=0}^{\infty}Q_{ij}^{(n)}(z,y)\bullet(1-{}^{A}H_j(t-z)),\end{aligned} \tag{3.32}$$

where the convolution product only acts on the temporal variable.
In case of conditional independence, this last result becomes:

$$M_{ij}(t,y)=\sum_{n=0}^{\infty}\int_0^t(1-{}^AH_j(t-z))d(p_{ij}\,{}^AF_{ij}(z)\,{}^BF_{ij}(y))^{(n)},$$
$$M_{ij}(t,y)=\sum_{n=0}^{\infty}(p_{ij}\,{}^AF_{ij}(z))^{(n)}({}^BF_{ij}(y))^{(n)}\bullet(1-{}^AH_j(t-z)). \tag{3.33}$$

Let us remark that for m=1, this last formula gives result (1.19) for Andersen's risk model.

3.1.4 The Premium Process

We will use the same approach as in section 1.1.2. From (3.29), we know that

$$\lim_{t\to\infty}\frac{{}^AH_{ij}(t)}{t}=\frac{\pi_j}{\sum_{k=1}^{m}\pi_k\eta_k},i,j\in I, \tag{3.34}$$

and it follows that for t large, we have:

$$E(N_j(t)|J_0=i)\approx\frac{t\pi_i}{\sum_{k=1}^{m}\pi_k\eta_k},i,j\in I, \tag{3.35}$$

and so, approximately, the mean cost of the $N_j(t)$ claims of type j on $(0,t]$ is

$$\frac{{}^B\eta_j}{\sum_k\pi_k\eta_k}t, \tag{3.36}$$

and finally, we get:

$$E(U_{N(t)}|J_0=i)\approx\frac{\sum_j\pi_j\,{}^B\eta_j}{\sum_k\pi_k\,{}^A\eta_k}. \tag{3.37}$$

This last relation shows that, whatever the initial state is, the total mean cost of the claims is more or less than $\tilde{c}t$ with

$$\tilde{c}\approx\frac{\sum_j\pi_j\,{}^B\eta_j}{\sum_k\pi_k\,{}^A\eta_k}. \tag{3.38}$$

It follows that if we take this value $\tilde{c}$ as *constant premium rate per unit of time*, we have an asymptotically fair game between the insurance company and the insured.

Let us point out that the value of this premium rate only depends on the means of inter-arrivals and claim amounts and also the stationary distribution of the embedded MC of successive claim or environment types. All of them may be easily computed with the statistical data of observed claims.

3.1.5 The Risk and Risk Reserve Processes

Definitions (1.20) and (1.21) are still valid for the SMRM so that the *risk process* is defined by $(U(t)-ct\ ,t\geq 0)$ and the *risk reserve process* by $\alpha=(\alpha(t),t\geq 0)$ with $\alpha(t)=u+U(t)\ -ct$ where u is as usual the *initial reserve* or *equity* of the company and c the *loaded premium rate*:

$$c=(1+\eta)\tilde{c}. \tag{3.39}$$

3.2 The Stationary Semi-Markov Risk Model

Using a result from Chapter 3 section 11, the stationary version of the SMRM is obtained if we take for (J_0,X_1) the following initial distributions:

$$\begin{aligned} &P(J_0=i)=\frac{\pi_i\,{}^A\eta_i}{\sum_k \pi_k\,{}^A\eta_k},\\ &P(X_1\leq x,J_1=j\,|\,J_0=i)=\frac{p_{ij}}{{}^A\eta_i}\int_0^x(1-{}^AF_{ij}(z)dz, \end{aligned} \tag{3.40}$$

so that

$$P(X_1\leq x,J_1=j)=\frac{\sum_i \pi_i p_{ij}\int_0^x(1-{}^AF_{ij}(z)dz}{\sum_k \pi_k\,{}^A\eta_k}. \tag{3.41}$$

Then we know that the process $((J_{N(t)},J_{N(t)+1},T_{N(t)+1}-t),t\geq 0)$ is stationary with:

$$P(J_{N(t)}=j,J_{N(t)+1}=k,T_{N(t)+1}-t\leq x)=\frac{\pi_j p_{jk}}{\sum_i \pi_i\,{}^A\eta_i}\int_0^x(1-\,{}^AF_{jk}(z)dz. \tag{3.42}$$

The interest of stationary models in classical risk theory (take here m=1!) has been investigated by Thorin (1975).

3.3 Particular SMRM With Conditional Independence

We know SMRM's with conditional independence are entirely characterized by the triplet of matrices $(\mathbf{P},{}^A\mathbf{F},{}^B\mathbf{F})$ where

$$\mathbf{P}=\left[p_{ij}\right],{}^A\mathbf{F}=\left[{}^AF_{ij}\right],{}^B\mathbf{F}=\left[{}^BF_{ij}\right], \tag{3.43}$$

as we know that in this case both claim arrival and claim amount processes are SMP characterized respectively by $(\mathbf{P},{}^A\mathbf{F})$ and $(\mathbf{P},{}^B\mathbf{F})$.

Paraphrasing, as before, Kendall's notation for queuing theory, we will denote this model by *SM/SM* with respect to the order claim arrival process/claim amount process.

3.3.1 The SM/G model

In this model, the claim arrival process is an SMP defined by the couple $(\mathbf{P}, {}^A\mathbf{F})$ and for ${}^B\mathbf{F}$, we choose

$$ {}^BF_{ij}(y) = B(y), \forall i, j \in I, y \geq 0. \tag{3.44}$$

This means that the sequence $(Y_n, n \geq 1)$ is a sequence of i.i.d. random variables with B as common d.f.

3.3.2 The G/SM model

This model is symmetric to the preceding one: the claim amount process is an SMP defined by the couple $(\mathbf{P}, {}^B\mathbf{F})$ and for ${}^A\mathbf{F}$, we choose

$$ {}^AF_{ij}(x) = A(x), \forall i, j \in I, x \geq 0. \tag{3.45}$$

This means that the sequence $(X_n, n \geq 1)$ is a sequence of i.i.d. random variables with A as common d.f.

3.3.3 The P/SM model

This model is a particular case of the *G/SM* model where in addition the claim arrival process is a Poisson process of parameter λ. It follows that

$$A(x) = \begin{cases} 0, x < 0, \\ 1 - e^{-\lambda x}, x \geq 0, \end{cases} \tag{3.46}$$

and of course that:

$$P(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \tag{3.47}$$

Remark 3.2 Let us remark that the intersection of the first two models, called the *G/G* model, gives Andersen's model with

$$m = 1, {}^AF_{ij}(x) = A(x), {}^BF_{ij}(y) = B(y), i, j \in I \tag{3.48}$$

and that the particular *P/SM* model with

$$m = 1, {}^BF_{ij}(y) = B(y), i, j \in I \tag{3.49}$$

is identical to the Cramer-Lundberg model (see section 1)

3.3.4 The M/SM model

Here, we keep a semi-Markov process $(\mathbf{P}, {}^{B}\mathbf{F})$ for the claim amount process and we assume that the claim arrival process is a continuous time Markov process (see Chapter 3, section 12.3), i.e. ${}^{A}\mathbf{F}$ is defined as:

$$ {}^{A}F_{ij}(x) = \begin{cases} 0, x < 0, \\ 1 - e^{-\lambda_i x}, x \geq 0. \end{cases} \tag{3.50}$$

3.3.5 The M'/SM Model

This is a variant of the preceding model, in which we set:

$$ {}^{A}F_{ij}(x) = \begin{cases} 0, x < 0, \\ 1 - e^{-\lambda_j x}, x \geq 0. \end{cases} \tag{3.51}$$

In this case, the interarrival time distribution only depends on the future state, and not on the past one, which is, as we already mentioned, more natural in risk theory, but now the arrival process is no longer Markovian.

3.3.6 The SM(0)/SM(0) Model

Particular assumptions of this model are:

$$ \begin{aligned} &{}^{A}F_{ij}(x) = {}^{A}F_{j}(x), \ {}^{B}F_{ij}(y) = {}^{B}F_{j}(y), i, j \in I, x, y > 0, \\ &p_{ij} = \pi_j, i, j \in I. \end{aligned} \tag{3.52}$$

So the claim types are independent and have the same distribution.
If we suppose that the initial type J_0 has the same distribution, the processes $(X_n, n \geq 1), (Y_n, n \geq 1)$ become two renewal processes having as d.f. respectively

$$ {}^{A}F(x) = \sum_j \pi_j \, {}^{A}F_j(x), \ {}^{B}F(y) = \sum_j \pi_j \, {}^{B}F_j(y), \tag{3.53}$$

but, due to the *J*-process, are not independent as:

$$ P(X_n \leq x, Y_n \leq y, J_n = j) = \pi_j \, {}^{A}F_j(x) \, {}^{B}F_j(y). \tag{3.54}$$

Let us recall that, in the terminology of Pyke (1962), the processes $((J_n, X_n), n \geq 0), ((J_n, Y_n), n \geq 0)$ are called *SMP of zero order of second type* (see Chapter 4, section 13.2).

3.3.7 The SM'(0)/SM'(0) Model

Particular assumptions of this model are:

$$ \begin{aligned} &{}^{A}F_{ij}(x) = {}^{A}F_{i}(x), \ {}^{B}F_{ij}(y) = {}^{B}F_{i}(y), i, j \in I, x, y > 0, \\ &p_{ij} = \pi_j, i, j \in I. \end{aligned} \tag{3.55}$$

Here too, the claim types are independent and have the same distribution but depending on the preceding claim.
If we suppose that the initial type J_0 has the same distribution, the processes $(X_n, n \geq 1), (Y_n, n \geq 1)$ become two renewal processes having as d.f. respectively

$$ {}^{A}F(x) = \sum_{j} \pi_j \, {}^{A}F_j(x), \; {}^{B}F(y) = \sum_{j} \pi_j \, {}^{B}F_j(y), \tag{3.56} $$

but, due to the *J*-process, not independent as:

$$ P(X_n \leq x, Y_n \leq y, J_n = j) = \pi_j \sum_{i} \pi_i \, {}^{A}F_i(x) \, {}^{B}F_i(y). \tag{3.57} $$

Let us recall that, in the terminology of Pyke (1962), the processes $((J_n, X_n), n \geq 0), ((J_n, Y_n), n \geq 0)$ are called *SMP of zero order of first type*.

3.3.8 The mixed zero order SM'(0)/SM(0) and SM(0)/SM'(0) models

These last two particular models are obtained by making the following two choices:
a) for the *SM'*(0)/*SM*(0) model:

$$ \begin{cases} p_{ij} = \pi_j, i, j \in I, \\ {}^{A}F_{ij}(x) = {}^{A}F_i(x), i \in I, x \geq 0, \\ {}^{B}F_{ij}(y) = {}^{B}F_j(y), j \in I, y \geq 0, \end{cases} \tag{3.58} $$

b) for the *SM*(0)/*SM'*(0) model:

$$ \begin{cases} p_{ij} = \pi_j, i, j \in I, \\ {}^{A}F_{ij}(x) = {}^{A}F_j(x), i \in I, x \geq 0, \\ {}^{B}F_{ij}(y) = {}^{B}F_i(y), j \in I, y \geq 0. \end{cases} \tag{3.59} $$

For the *SM'*(0)/*SM*(0) model, it is straightforward to see that:

$$ P(X_n \leq x, Y_n \leq y, J_n = j) = \pi_j \, {}^{B}F_j(y) \sum_{i} \pi_i \, {}^{A}F_i(x) \tag{3.60} $$

and for the *SM*(0)/*SM'*(0) model:

$$ P(X_n \leq x, Y_n \leq y, J_n = j) = \pi_j \, {}^{A}F_j(x) \sum_{i} \pi_i \, {}^{B}F_i(y). \tag{3.61} $$

Here too, the processes $(X_n, n \geq 1), (Y_n, n \geq 1)$ become two renewal processes having the d.f given by relation (3.56) but furthermore they are independent.
Consequently, if one makes abstractions of the types of claims, these two last models may be treated, at least for the ruin problem, as a *G/G* model characterised by the d.f. ${}^{A}F, {}^{B}F$.

3.4 The Ruin Problem For The General SMRM

3.4.1 Ruin and Non-Ruin Probabilities

Using definition (1.22) for the lifetime T of the company,

$$T = \inf\{t : \alpha(t) < 0\}, \tag{3.62}$$

we know that the event "*ruin*" occurs before or at time t iff $T \le t$ and of course the complementary event called "*non-ruin*" iff $T>t$.
As we must now take into account the types of claims, we will use the following notation for *transient non-ruin* and *ruin probabilities,* i.e. *on the finite time horizon* $[0,t]$,

$$\begin{aligned} \phi_{ij}(u,t) &= P(T > t, Z(t) = j \mid Z(0) = i), \\ \Psi_{ij}(u,t) &= P(T \le t, Z(t) = j \mid Z(0) = i)(=1-\varphi_{ij}(u,t)). \end{aligned} \tag{3.63}$$

The asymptotic non-ruin and ruin probabilities, i.e. *on an infinite time horizon*, are defined as

$$\begin{aligned} \phi_{ij}(u) &= P(T = \infty, Z(t) = j \mid Z(0) = i) = \lim_{t\to\infty} \phi_{ij}(u,t), \\ \Psi_{ij}(u) &= P(T < \infty, Z(t) = j \mid Z(0) = i) = \lim_{t\to\infty} \Psi_{ij}(u,t)(=1-\varphi_{ij}(u)). \end{aligned} \tag{3.64}$$

The following results are trivial but useful:

(i) for every fixed $t, \forall i, j \in I, \phi_{ij}(u,t)$ is increasing in u and $\Psi_{ij}(u,t)$ decreasing,

(ii) for every fixed $u, \forall i, j \in I, \phi_{ij}(u,t)$ is decreasing in t and $\Psi_{ij}(u,t)$ increasing,

(iii) $\forall u, \forall t, \forall i, j \in I : \phi_{ij}(u,t) \ge \phi_{ij}(u), \Psi_{ij}(u,t) \ge \Psi_{ij}(u).$ (3.65)

As we already said in section 1.1.4, one of the most important problems in risk theory is the optimal determination of the security loading η such that the probability of ruin, transient or asymptotic, is larger than $1-\varepsilon, \varepsilon > 0$ being fixed, a problem equivalent to the optimal determination of the *solvency margin*.
Often, the problem is solved for the stationary version, thus giving excessive and therefore careful values for η.
As is often the case, if we are not interested in the value of $Z(t)$, then we may introduce the following ruin probabilities:

$$\begin{aligned} \phi_i(u,t) &= \sum_{j=1}^{m} \phi_{ij}(u,t), \phi_i(u) = \sum_{j=1}^{m} \phi_{ij}(u), \\ \Psi_i(u,t) &= \sum_{j=1}^{m} \Psi_{ij}(u,t), \Psi_i(u) = \sum_{j=1}^{m} \Psi_{ij}(u). \end{aligned} \tag{3.66}$$

Moreover, if we start with π as initial distribution for J_0, we define the last four ruin and non-ruin probabilities:

$$\begin{aligned}\phi(u,t)&=\sum_{i=1}^{m}\pi_i\phi_i(u,t),\phi(u)=\sum_{i=1}^{m}\pi_i\phi_i(u),\\ \Psi\ (u,t)&=\sum_{i=1}^{m}\pi_i\Psi_i(u,t),\Psi(u)=\sum_{i=1}^{m}\pi_i\Psi_i(u).\end{aligned}\tag{3.67}$$

Finally, for the stationary model defined in section 3.2, we introduce these last ruin and non-ruin probabilities:

$$\begin{aligned}&{}^{s}\phi_j(u)=\frac{1}{\sum_k \pi_k\eta_k}\sum_l\sum_{l'}\int_0^{\infty}(1-{}^{A}F_{ll'}(t)dt\int_0^{u+ct}\phi_{l'j}(u+ct-y)d\,{}^{B}F_{ll'}(y),\\ &{}^{s}\phi(u)=\sum_{j=1}^{m}{}^{s}\phi_j(u),{}^{s}\Psi(u)=1-{}^{s}\phi(u).\end{aligned}\tag{3.68}$$

3.4.2 Change of Premium Rate

Let us start with a general SMRM of kernel $\mathbf{Q}=\left[Q_{ij}(.,.)\right]$ and with c as premium rate per unit of time.
First, we will show that, without loss of generality, we can always work with an equivalent SMRM for which c=1.
Indeed, if such is not the case, let us introduce the following new r.v.:

$$X_0^{'}=X_0,X_n^{'}=cX_n,n\geq 1\text{ ,a.s.}\tag{3.69}$$

The SM kernel $\mathbf{Q}'$ of the process $\left(J_n,X_n^{'},Y_n,n\geq 0\right)$ is given by:

$$\begin{aligned}&Q_{J_{n-1}j}^{'}(x,y)=P(J_n=j,X_n^{'}\leq x,Y_n\leq y\left|(J_k,X_k^{'},Y_k,),k=1,...,n-1\right.),\\ &Q_{J_{n-1}j}^{'}(x,y)=Q_{J_{n-1}j}(\frac{x}{c},y).\end{aligned}\tag{3.70}$$

For the SMRM with kernel $\mathbf{Q}'$, we have:

$$p_{ij}^{'}=p_{ij},a_{ij}^{'}=ca_{ij},b_{ij}^{'}=b_{ij},i,j\in I\tag{3.71}$$

so that

$${}^{A}\eta_i^{'}=c\,{}^{A}\eta_i,{}^{B}\eta_i^{'}=c\,{}^{B}\eta_i,i\in I,\tag{3.72}$$

and by relation (3.20)

$$\mu_{jj}^{'}=c\mu_{jj},j\in I,\tag{3.73}$$

and by relations (1.10) and (3.38)

$$\tilde{c}'=\frac{1}{c}\frac{\sum_i\pi_i{}^{B}\eta_i}{\sum_i\pi_i{}^{A}\eta_i}=\frac{\tilde{c}}{c}.\tag{3.74}$$

Taking into account the loading factor η, by relation (3.39), we know that the value of c' is given by:

$$c' = (1+\eta)\tilde{c}' \tag{3.75}$$

or by (3.74) and(3.39) again:

$$c' = (1+\eta)\frac{\tilde{c}}{c} = 1. \tag{3.76}$$

As in this last relation the loading factor η is strictly positive, the last equality shows that the ratio $\tilde{c}/c$ is strictly inferior to 1 and so by relations (3.74) and (3.72) that the condition of having a strictly positive loading factor η is equivalent to having, for the process $\left(J_n, X_n^{'}, Y_n, n \geq 0\right)$, the condition

$$\frac{\sum_i \pi_i {}^B\eta_i}{\sum_i \pi_i {}^A\eta_i^{'}} < 1. \tag{3.77}$$

3.4.3 General Solution Of The Asymptotic Ruin Probability Problem for a general SMRM

From the results of the last subsection, let us consider a general ergodic SMRM of kernel $\mathbf{Q} = \left[Q_{ij}(.,.)\right]$ with c=1 and let us focus our attention on the process

$$\left((J_n, Y_n - X_n), n \geq 0\right); \tag{3.78}$$

Clearly, this process can be seen as defining an SMRW of SM kernel ${}^r\mathbf{Q}$ given by

$${}^rQ_{ij}(z) \quad \iint_{\{(\xi,\zeta):\xi-\zeta\leq z\}} Q_{ij}(d\xi, d\zeta). \tag{3.79}$$

In the special case of conditional independence, we get:

$${}^rQ_{ij}(z) = p_{ij}\int_{-\infty}^{+\infty} {}^BF_{ij}(z+\xi)d\,{}^AF_{ij}(\xi). \tag{3.80}$$

We know that the position at epoch n of the SMRW is given by the partial sums $(S_n, n \geq 0)$ related to the random sequence $((Y_n - X_n, n \geq 0)$ and introducing the r.v. M defined by relation (20.1) of Chapter 3, that is $M = \sup\left\{S_0, S_1, ..., S_n, ...\right\}$, we know that:

$$\phi_{ij}(u) = P(M \leq u, \lim_{n\to\infty} J_n = j \,|\, J_0 = i), u \geq 0, i, j \in I, \tag{3.81}$$

and since the event non-ruin on $\left[0,\infty\right)$ implies non-ruin at the first claim arrival time, we have:

$$\phi_{ij}(u)=\begin{cases}0, u<0,\\ \sum_k \int_{-\infty}^{u}\phi_{kj}(u-z)d\,{}^rQ_{ik}(z), u>0.\end{cases}, i,j\in I \tag{3.82}$$

Summing over j, we get:

$$\phi_{i}(u)=\begin{cases}0, u<0,\\ \sum_k \int_{-\infty}^{u}\phi_{k}(u-z)d\,{}^rQ_{ik}(z), u>0,\end{cases} i,\in I. \tag{3.83}$$

which is a *Wiener-Hopf system of integral equations* identical to the system (8.6) of Chapter 5 as

$$\phi_i(u)=P(M\le u,|J_0=i), u\ge 0, i\in I \tag{3.84}$$

as in relation (20.6) of Chapter 3.
Considering now system (3.83) only for non-negative values of u, we know from a result of Janssen (1970) mentioned in section 20 of Chapter 3 that this system has a unique **P**-solution iff

$$\sum_k \pi_k\,{}^r\eta_k<0. \tag{3.85}$$

So, if this condition is not fulfilled, the SMRW $\left((J_n,S_n),n\ge 0\right)$ drifts towards $+\infty$ and so ruin on $\left[0,\infty\right)$ is a certain event regardless of J_0, that is, in this case:

$$\phi_i(u)=0, u\ge 0, i\in I. \tag{3.86}$$

As

$${}^r\eta_k={}^B\eta_k-{}^A\eta_k, k\in I, \tag{3.87}$$

condition (3.83) is equivalent to

$$\sum_k \pi_k({}^B\eta_k-{}^A\eta_k)<0, \tag{3.88}$$

which is equivalent to condition (3.77) always supposed to be fulfilled and equivalent to the adjunction of a positive security loading to spoil the asymptotically fair game "insurance company-policyholders" in favour of the insurance company as, without this adjunction the ruin on an infinite horizon time is certain.
Using the unicity theorem of Janssen (1970), it is clear that the following relations are true:

$$\phi_{ij}(u)=\pi_j\phi_i(u), i,j\in I, u\ge 0. \tag{3.89}$$

We have thus proved the following theorem.

Theorem 3.1 *For an ergodic SMRM, for every initial state i, the ruin on an infinite time horizon is certain if condition* (3.77) *is not fulfilled.*
On the contrary, if this condition is satisfied, then for every initial state i:

$$\phi_{ij}^{+}(u)=\pi_j\phi_i^{+}(u), i,j\in I, u\in\mathbb{R}, \tag{3.90}$$

with

$$\phi_{ij}^{+}(u) = U_0(u)\phi_{ij}(u), \phi_i^{+}(u) = \phi_i(u), i, j \in I, u \in \mathbb{R} \quad (3.91)$$

and by (3.81)

$$\phi_i^{+}(u) = \sum_k \int_{-\infty}^{u} \phi_k^{+}(u-z) d\,{}^rQ_{ik}(z), u \in \mathbb{R}, i, \in I. \quad (3.92)$$

Let us mention that the passage to the functions ϕ_{ij}^{+} is done because we are only interested in the non-ruin probabilities for positive values of the reserve *u,* though it may be that these probabilities are not necessarily zero for *u* negative.
Though there exist a lot of theoretical results on the Wiener-Hopf integral system (3.92) like factorisation results and so on, there exist no explicit forms for non-ruin probabilities for a general SMRM and so our approach is to see what particular SMRM can be treated in such a way that right information can be obtained by transferring the problem towards a model for which an explicit solution exists or at least having satisfactory information.
Of course, for the general SMRM, the numerical approach remains very important.

Remark 3.3 We know that for *m*=1, the GSMRM brings us back to the *G/G* model.
So, for this one, the system (3.92) becomes:

$$\begin{aligned} &\phi^{+}(u) = \int_{-\infty}^{u} \phi^{+}(u-z) d\,{}^rQ(z), u \in \mathbb{R}, \\ &{}^rQ(z) = \int_{-\infty}^{+\infty} B(\xi + u) dA(\xi). \end{aligned} \quad (3.93)$$

3.5 The Ruin Problem For Particular SMRM

This section undertakes an analytical study of the particular SMRM introduced above to get supplementary results for solving the ruin problem in these particular cases.

3.5.1 The Zero Order Model SM(0)/SM(0)

In this case, we have from relation (3.79) that:

$${}^rQ_{ij}(z) = \pi_j \int_{-\infty}^{+\infty} {}^BF_j(\xi + z) d\,{}^AF_j(\xi), \quad (3.94)$$

such that we may write:

$${}^rQ_{ij}(z) = {}^rQ_j(z) \quad (3.95)$$

and the Wiener-Hopf system (3.92) becomes:

$$\phi_i^{+}(u) = \sum_k \int_{-\infty}^{u} \phi_k^{+}(u-z) d\,{}^rQ_k(z), u \in \mathbb{R}, i, \in I. \quad (3.96)$$

It follows that all the functions ϕ_i^+ i=1,...,m are equal to a common function ϕ^+ satisfying (3.96), that is:

$$\begin{aligned}\phi^+(u) &= \int_{-\infty}^{u} \phi^+(u-z)d\,{}^rQ(z), u \in \mathbb{R},\\ {}^rQ(z) &= \sum_k {}^rQ_k(z) = \sum_k \pi_j \int_{-\infty}^{+\infty} {}^BF_k(\xi+z)d\,{}^AF(\xi), z \in \mathbb{R}.\end{aligned} \tag{3.97}$$

In conclusion, relation (3.97) shows that *the non-ruin probabilities for zero order models SM*(0)/*SM*(0) *can be computed by an associated G/G model of kernel* ${}^r\mathbf{Q}$.

3.5.2 The Zero Order Model SM'(0)/SM'(0)

In this case, we have from relation (3.79) that:

$$\begin{aligned}{}^rQ_{ij}(z) &= \pi_j \int_{-\infty}^{+\infty} {}^BF_i(\xi+z)d\,{}^AF_i(\xi)\\ &= \pi_j F_i(z)\end{aligned} \tag{3.98}$$

with

$$F_i(z) = \int_{-\infty}^{+\infty} {}^BF_i(\xi+z)d\,{}^AF_i(\xi), i = 1,...,m, z \in \mathbb{R} \tag{3.99}$$

such that the Wiener-Hopf system (3.92) becomes:

$$\begin{aligned}\phi_i^+(u) &= \sum_k \pi_k \int_{-\infty}^{u} \phi_k^+(u-z)dF_i(z), u \in \mathbb{R}\\ &= \int_{-\infty}^{u} (\sum_k \pi_k \phi_k^+(u-z))dF_i(z).\end{aligned} \tag{3.100}$$

From this last relation, it follows that, using definition (3.67), we get:

$$\begin{aligned}\phi^+(u) &= \int_{-\infty}^{u} \phi^+(u-z)d\,{}^rQ(z), u \in \mathbb{R},\\ {}^rQ(z) &= \sum_k \pi_k \int_{-\infty}^{+\infty} {}^BF_k(\xi+z)d\,{}^AF_k(\xi), z \in \mathbb{R}.\end{aligned} \tag{3.101}$$

If we suppose to know the value of ϕ^+, the non-ruin probabilities ϕ_i^+ are also known from relation (3.100):

$$\phi_i^+(u) = \int_{-\infty}^{u} \phi^+(u-z)dF_i(z), i = 1,...,m. \tag{3.102}$$

In conclusion, relation (3.101) and (3.102) show that the non-ruin probabilities for zero order models *SM*'(0)/*SM*'(0) can be computed by an associated *G/G* model of kernel ${}^r\mathbf{Q}$.

3.5.3 The Model M/SM

Supposing that claim distributions depend only on the type of the occurred claim, let us recall that for this model, we have:

$$ {}^{A}F_{ij}(x) = \begin{cases} 0, x \le 0, \\ 1 - e^{-\lambda_i x}, x \ge 0, \end{cases} \quad {}^{B}F_{ij}(y) = B_j(y), i, j \in I. \tag{3.103} $$

Instead of starting with the Wiener-Hopf system, we may resort to elementary probabilistic reasoning to obtain the following relations:

$$ \phi_i(u) = \sum_{j=1}^{m} p_{ij} \int_0^{\infty} \lambda_i e^{-\lambda_i \tau} d\tau \int_0^{u+\tau} \phi_j(u + \tau - x) dB_j(x), i \in I. \tag{3.104} $$

These relations are obtained by conditioning with the occurrence of the first claim that occurs in $(t,t+dt)$ and of type j.
Using the change of variables $u + \tau = \xi$, we get:

$$ \phi_i(u) = \sum_{j=1}^{m} p_{ij} \int_u^{\infty} \lambda_i e^{-\lambda_i (\xi - u)} d\xi \int_0^{\xi} \phi_j(\xi - x) dB_j(x), i \in I. \tag{3.105} $$

These last relations show that the non-ruin probabilities ϕ_i are differentiable and that:

$$ \phi_i'(u) = \phi_i(u) - \lambda_i \sum_{j=1}^{m} p_{ij} \int_0^{\xi} \phi_j(\xi - x) dB_j(x), i \in I. \tag{3.106} $$

To simplify, let us suppose that the d.f. B_j are differentiable with b_j as derivative; then, taking Laplace transforms of both members of this relation, we get (using notation introduced in Chapter 2, section 5.1):

$$ s\tilde{\phi}_i(s) - \phi_i(0) = \lambda_i \tilde{\phi}_i(s) - \lambda_i \sum_{j=1}^{m} p_{ij} \tilde{\phi}_j(s) \tilde{b}_j(s), i \in I \tag{3.107} $$

or

$$ s\tilde{\phi}_i(s) + \lambda_i \sum_{j=1}^{m} (p_{ij} \tilde{b}_j(s) - \delta_{ij}) \tilde{\phi}_j(s) = \phi_i(0), i \in I. \tag{3.108} $$

With the following matrix notation:

$$ \tilde{\mathbf{M}}(s) = \left[\lambda_i p_{ij} \tilde{b}_j(s) \right], \mathbf{\Lambda} = \left[\lambda_i \delta_{ij} \right], \tilde{\mathbf{b}}(s) = \left[\delta_{ij} \tilde{b}_j(s) \right], \tag{3.109} $$

and if $\tilde{\phi}(s)$ and $\phi(t)$ represent respectively the column vectors of the functions $\tilde{\phi}_i(s), i = 1,...,m$ and $\phi_i(t), i = 1,...,m$, the last relation may be written in the form:

$$ s\tilde{\phi} + (\tilde{\mathbf{M}} - \mathbf{\Lambda})\tilde{\phi} = \phi(0), \tag{3.110} $$

and as

$$ \tilde{\mathbf{M}} = \mathbf{\Lambda P} \tilde{\mathbf{b}}, \tag{3.111} $$

we get:

$$ s\left(\mathbf{I} - \mathbf{\Lambda} \frac{(\mathbf{I} - \mathbf{P}\tilde{\mathbf{b}})}{s} \right) \tilde{\phi} = \phi(0). \tag{3.112} $$

With the matrix norm defined as the sum of the absolute values of all its elements, the norm of the matrix $\tilde{\mathbf{A}}(s)$ defined as

$$\tilde{\mathbf{A}}(s) = \frac{1}{s}\boldsymbol{\Lambda}(\mathbf{I} - \mathbf{P}\tilde{\mathbf{b}}(s)) \tag{3.113}$$

is strictly inferior to 1 for s sufficiently large as a result of the fact that

$$\lim_{s\to\infty} \tilde{\mathbf{A}}(s) = \mathbf{0}. \tag{3.114}$$

Consequently, for such values of s, $\mathbf{I} - \tilde{\mathbf{A}}(s)$ is invertible and moreover

$$(\mathbf{I} - \tilde{\mathbf{A}}(s))^{-1} = \sum_{n=0}^{\infty} \left(\tilde{\mathbf{A}}(s)\right)^n. \tag{3.115}$$

By the inverse Laplace transform, relation (3.113) gives the value of the matrix **A**:

$$\mathbf{A}(t) = \boldsymbol{\Lambda}(\mathbf{I} - \mathbf{P}\mathbf{B}(t)) \tag{3.116}$$

where **B***(t)* is the diagonal matrix

$$\begin{aligned} &\mathbf{B}(t) = (\delta_{ij} B_j(t)), \\ &\mathbf{B}_j(t) = \int_0^t b_j(z)dz, j = 1,...,m. \end{aligned} \tag{3.117}$$

By the inverse Laplace transform again, from relations (3.112), (3.116) and (3.115), we find a theoretical explicit expression for the vector φ,

$$\begin{aligned} &\phi(u) = \left(\sum_{nj=0}^{\infty} \int_0^u \mathbf{A}^{(n)}(t)dt\right)\phi(0), \\ &\phi(0) = \left(\sum_{nj=0}^{\infty} \int_0^u \mathbf{A}^{(n)}(t)dt\right)^{-1} \mathbf{1} \end{aligned} \tag{3.118}$$

where $\mathbf{1}$ is the m-dimensional vector with all components equal to1 as $\lim_{u\to\infty} \phi(u) = \mathbf{1}$.

Remark 3.4

(i) For m=1, we get the result (1.60) for the *P/G* model.

(ii) For the stationary model introduced in section 3.2, we know from relation (3.68) that

$$^s\phi(u) = \frac{1}{\sum_k \pi_k \lambda_k^{-1}} \sum_l \sum_{l'} \pi_l p_{ll'} \int_0^{\infty} e^{-\lambda_l z} dz \int_0^{u+z} \phi_{l'}(u+z-x) dB_{l'}(x)\,, \tag{3.119}$$

so that from relation (3.105), we get:

$$^s\phi(u) = \frac{1}{\sum_k \pi_k \lambda_k^{-1}} \sum_l \frac{\pi_l}{\lambda_l} \phi_l(u). \tag{3.120}$$

If all the λ_k are equal to λ, we have a *P/SM* model for which the last relation becomes:

$$ {}^{s}\phi(u) = \sum_{l} \pi_l \phi_l(u). \tag{3.121} $$

(iii) If $c \neq 1,$ it suffices to replace in the above formulas, λ_i by $\lambda_i / c, i = 1,...,m.$

3.5.4 The Zero Order Models As Special Case Of The Model M/SM

3.5.4.1 The P/SM(0) model

In section 3.5.1, for the Zero Order Model SM(0)/SM(0) we established the equality of all the non-ruin probabilities $\phi_i, i = 1,...,m.$
As for the *P/SM*(0) model, we have:

$$ {}^{A}F(x) = {}^{A}F(x) = \begin{cases} 0, x < 0, \\ 1 - e^{-\lambda x}, x \geq 0, \end{cases} \tag{3.122} $$

we know from relation (3.97) that:

$$ {}^{r}Q(u) = \int_{-\infty}^{+\infty} (\sum_{j} \pi_j \, {}^{B}F_j(\xi + z) d\, {}^{A}F(\xi) \tag{3.123} $$

and consequently, the *G/G* associated model becomes here a *P/G* model with

$$ B(y) = \sum_{j} \pi_j \, {}^{B}F_j(y). \tag{3.124} $$

So, from the result (1.60), we obtain

$$ \begin{aligned} &\phi_i(u) = \phi(u) = (1 - \lambda\beta) \sum_{n=0}^{\infty} \int_0^u \overline{B}^{(n)}(t) dt, i = 1,...,m, \\ &\overline{B}(t) = 1 - \sum_{j} \pi_j \, {}^{B}F_j(t), \\ &\beta = \sum_{j} \pi_j \, {}^{B}\eta_j. \end{aligned} \tag{3.125} $$

For the stationary model, we get from relation (3.121) that:

$$ {}^{s}\phi(u) = \phi(u). \tag{3.126} $$

3.5.4.2 The P/SM'(0) model

The *P/SM'*(0) model is the particular *SM'*(0)/*SM'*(0) model for which (3.122) still holds.
By relation (3.101), we know that the function ϕ_i is still given by the expression (3.125) and that for the non-ruin probabilities, we have result (3.102),

$$ \phi_i^{+}(u) = \int_{-\infty}^{u} \phi^{+}(u - z) dF_i(z), i = 1,...,m \tag{3.127} $$

with, by (3.99):

$$F_i(z) = \lambda \int_0^{+\infty} {}^B F_i(\xi + z) e^{-\lambda \xi} d\xi, i = 1,...,m, z \in \mathbb{R}. \quad (3.128)$$

3.6 The M'/SM Model

3.6.1 General solution

Let us recall that for this model, we have:

$${}^A F_{ij}(x) = \begin{cases} 0, x \le 0, \\ 1 - e^{-\lambda_j x}, x \ge 0, \end{cases} \quad {}^B F_{ij}(y) = B_j(y), i, j \in I. \quad (3.129)$$

Let us first verify that the process $((J_{n+1}, X_n, Y_n,), n \ge 0)$ is an SMRP satisfying the assumption of conditional independence. Indeed, we may write that

$$P(J_{n+1} = l, X_n \le x, Y_n \le y \,|\, (J_\nu, X_\nu, Y_\nu), \nu \le n, J_n = k) = p_{kl} \, {}^A F_k(x) B_k(y). \quad (3.130)$$

Now let $\overline{\phi}_i(u), i = 1,...,m$ be the non-ruin probabilities related to the process $((J_{n+1}, X_n, Y_n,), n \ge 0)$ starting with $J_1 = i$.

It is clear that the probabilities we want to know, $\phi_i(u), i = 1,...,m$, are given by:

$$\phi_i(u) = \sum_{j=1}^m p_{ij} \overline{\phi}_j(u), i = 1,...,m. \quad (3.131)$$

So, it suffices to know the non-ruin probabilities $\overline{\phi}_i(u), i = 1,...,m$.

Using a reasoning similar to that used for the *M/SM*(0) model, we get:

$$\overline{\phi}_i(u) = \sum_{j=1}^m p_{ij} \int_0^\infty \lambda_i e^{-\lambda_i \tau} d\tau \int_0^{u+\tau} \overline{\phi}_j(u + \tau - x) dB_i(x), i \in I. \quad (3.132)$$

These relations are obtained by conditioning with the occurrence of the first claim that occurs in (*t,t+dt*) and of type *j*.

Using the change of variables $u + \tau = \xi$, we get:

$$\overline{\phi}_i(u) = \sum_{j=1}^m p_{ij} \int_u^\infty \lambda_i e^{-\lambda_i(\xi - u)} d\xi \int_0^\xi \overline{\phi}_j(\xi - x) dB_i(x), i \in I. \quad (3.133)$$

These last relations show that the non-ruin probabilities ϕ_i are differentiable and that:

$$\overline{\phi}_i'(u) = \lambda_i \overline{\phi}_i(u) - \lambda_i \sum_{j=1}^m p_{ij} \int_0^\xi \overline{\phi}_j(\xi - x) dB_i(x), i \in I. \quad (3.134)$$

This system is similar to (3.106) and here too, to simplify, let us suppose that the d.f. B_j are differentiable with b_j as derivative; then, taking Laplace transforms of both members of this relation, we get (using notation introduced in Chapter 2, section 5.1):

$$s\tilde{\bar{\phi}}_i(s)-\bar{\phi}_i(0)=\lambda_i\tilde{\bar{\phi}}_i(s)-\lambda_i\sum_{j=1}^{m}p_{ij}\tilde{\bar{\phi}}_j(s)\tilde{b}_i(s), i\in I, \tag{3.135}$$

or

$$s\tilde{\bar{\phi}}_i(s)+\lambda_i\sum_{j=1}^{m}(p_{ij}\tilde{b}_j(s)-\delta_{ij})\tilde{\bar{\phi}}_j(s)=\bar{\phi}_i(0), i\in I. \tag{3.136}$$

With the following matrix notation:

$$\tilde{\mathbf{N}}(s)=\left[\lambda_i p_{ij}\tilde{b}_j(s)\right], \mathbf{\Lambda}=\left[\lambda_i\delta_{ij}\right], \tilde{\mathbf{b}}(s)=\left[\delta_{ij}\tilde{b}_i(s)\right], \tag{3.137}$$

and with $\tilde{\bar{\phi}}(s)$ and $\bar{\phi}(t)$ representing respectively the column vectors of the functions $\tilde{\bar{\phi}}_i(s), i=1,...,m$ and $\bar{\phi}_i(t), i=1,...,m$, the last relation may be written in the form:

$$s\tilde{\bar{\phi}}+(\tilde{\mathbf{N}}-\mathbf{\Lambda})\tilde{\bar{\phi}}=\bar{\phi}(0), \tag{3.138}$$

and as

$$\tilde{\mathbf{N}}=\mathbf{\Lambda}\tilde{\mathbf{b}}\mathbf{P}, \tag{3.139}$$

we get:

$$s\left(\mathbf{I}-\mathbf{\Lambda}\frac{(\mathbf{I}-\tilde{\mathbf{b}}\mathbf{P})}{s}\right)\tilde{\bar{\phi}}=\bar{\phi}(0). \tag{3.140}$$

As in section 3.4, with the matrix norm defined as the sum of the absolute values of all its elements, the norm of the matrix $\tilde{\underline{\mathbf{A}}}(s)$ defined as

$$\tilde{\underline{\mathbf{A}}}(s)=\frac{1}{s}\mathbf{\Lambda}(\mathbf{I}-\tilde{\mathbf{b}}(s)\mathbf{P}) \tag{3.141}$$

is strictly inferior to 1 for s sufficiently large as a result of the fact that

$$\lim_{s\to\infty}\tilde{\underline{\mathbf{A}}}(s)=\mathbf{0}. \tag{3.142}$$

Consequently, for such values of s, $\mathbf{I}-\tilde{\underline{\mathbf{A}}}(s)$ is invertible and moreover

$$(\mathbf{I}-\tilde{\underline{\mathbf{A}}}(s))^{-1}=\sum_{n=0}^{\infty}\left(\tilde{\underline{\mathbf{A}}}(s)\right)^n. \tag{3.143}$$

By the inverse Laplace transform, relation (3.141) gives the value of the matrix $\underline{\mathbf{A}}$:

$$\underline{\mathbf{A}}(t)=\mathbf{\Lambda}(\mathbf{I}-\mathbf{B}(t)\mathbf{P}) \tag{3.144}$$

where $\mathbf{B}(t)$ is the diagonal matrix

$$\begin{aligned}\mathbf{B}(t)&=\left[\delta_{ij}B_j(t)\right],\\ B_j(t)&=\int_0^t b_j(z)dz, j=1,...,m.\end{aligned} \tag{3.145}$$

By the inverse Laplace transform again, from relations (3.142), (3.144) and (3.143), we find a theoretical explicit expression for the vector $\bar{\phi}$,

$$\begin{aligned}\overline{\phi}(u) &= \left(\sum_{nj=0}^{\infty}\int_0^u \underline{\mathbf{A}}^{(n)}(t)dt\right)\overline{\phi}(0),\\ \overline{\phi}(0) &= \left(\sum_{nj=0}^{\infty}\int_0^u \underline{\mathbf{A}}^{(n)}(t)dt\right)^{-1}\mathbf{1}\end{aligned} \tag{3.146}$$

where $\mathbf{1}$ is the m-dimensional vector with all components equal to 1 as

$$\lim_{u\to\infty}\overline{\phi}(u) = \mathbf{1}. \tag{3.147}$$

Together with relation (3.130), relations (3.147) are formally equivalent to relations (3.118) for the M/SM(0) model for giving the explicit form of ruin probabilities for the M'/SM(0) model. The difference between these two models appears in the definitions of the matrices **A** and $\underline{\mathbf{A}}$ given by relations (3.118) and (3.144).

The non-ruin probabilities for the model *M/SM*(0) and the non-ruin probabilities $\overline{\phi}_i(u), i = 1,...,m$ for the model *M'/SM*(0) are equal if the matrices **P** and **B**(*t*) commute for all positive *t* which is certainly true for *m*=1. For *m*>1, **P** and **B**(*t*) commute for all positive *t* iff functions B_i, B_j are identical whenever $p_{ij} > 0$, i.e. whenever states *i* and *j* communicate in one step in the embedded MC of the types of claims.

Remark 3.5

(i) For the stationary model introduced in section 3.2, we know from relation (3.68) that

$${}^{s}\phi(u) = \frac{1}{\sum_k \pi_k \lambda_k^{-1}} \sum_l \sum_{l'} \pi_l p_{ll'} \int_0^{\infty} e^{-\lambda_{l'} z} dz \int_0^{u+z} \phi_{l'}(u+z-x)dB_{l'}(x). \tag{3.148}$$

From relations (3.131) and (3.133), we get

$$\overline{\phi}_i(u) = \lambda_i \int_0^{\infty} e^{-\lambda_i z} dz \int_0^{u+z} \phi_i(u+z-x)dB_i(x) \tag{3.149}$$

which are inverse relations of (3.131).

Using these relations in (3.148), we get the relation analogous to (3.120),

$$\begin{aligned}{}^{s}\phi(u) &= \frac{1}{\sum_k \pi_k \lambda_k^{-1}} \sum_l \sum_{l'} \pi_l p_{ll'} \frac{\overline{\phi}_{l'}(u)}{\lambda_{l'}}\\ &= \frac{\sum_{l'} \pi_{l'} \lambda_{l'}^{-1} \overline{\phi}_{l'}(u)}{\sum_k \pi_k \lambda_k^{-1}}.\end{aligned} \tag{3.150}$$

(ii) If $c \neq 1$, it suffices to replace in the above formulas, λ_i by $\lambda_i / c, i = 1,...,m.$

3.6.2 Particular cases: the M/M and M'/M models

3.6.2.1 The M/M model

The *M/M* model is the particular *M/SM* model for which:

$$ {}^{B}F_j(y)=\begin{cases} 0, y\le 0, \\ 1-e^{-\frac{y}{\beta_j}}, y\ge 0. \end{cases} \tag{3.151}$$

In this case, Janssen and Reinhard (1985) proved that the non-ruin probabilities $\phi_i, i=1,...,m$ can be written in the form:

$$\phi_i(u)=\sum_j p_{ij}\int_0^\infty \lambda_i e^{-\lambda_i t}L_j(u+t)dt, u\ge 0, i=1,...,m \tag{3.152}$$

where the functions L_j are solutions of the differential system:

$$\begin{aligned} &L_j''(u)+(\beta_j^{-1}-\lambda_j)L_j'(u)-\frac{\lambda_j}{\beta_j}(L_j(u)-\sum_k p_{jk}L_k(u))=0, \\ &L_j(0)=0, L_j(\infty)=1, j=1,...,m, u\ge 0. \end{aligned} \tag{3.153}$$

For *m*=2, they give the following explicit form:

$$\phi_j(u)=1-\alpha_{j1}e^{k_{j1}u}-\alpha_{j2}e^{k_{j2}u}, j=1,2 \tag{3.154}$$

with an explicit form of the coefficients.

3.6.2.2 The M'/M model

Here too, the *M'/M* model is the particular *M'/SM* model for which:

$$ {}^{B}F_j(y)=\begin{cases} 0, y\le 0, \\ 1-e^{-\frac{y}{\beta_j}}, y\ge 0. \end{cases} \tag{3.155}$$

Here, relations (3.123) become:

$$\bar{\phi}_i'(u)=\lambda_i\bar{\phi}_i(u)-\frac{\lambda_i}{b_j}\sum_j p_{ij}\int_0^u \bar{\phi}_j(z)e^{-\frac{u-z}{b_j}}dz, i\in I. \tag{3.156}$$

It follows that the functions $\bar{\phi}_i, i=1,...,m$ are twice differentiable and that:

$$\begin{aligned} \bar{\phi}_i''(u)&=\lambda_i\bar{\phi}_i'(u)-\lambda_i\sum_j p_{ij}\left[-\frac{1}{b_j^2}\int_0^u \bar{\phi}_j(z)e^{-\frac{u-z}{b_j}}dz+\frac{1}{b_j}\bar{\phi}_j(u)\right] \\ &=\lambda_i\bar{\phi}_i'(u)-\frac{1}{b_j}\left[\lambda_i\bar{\phi}_i(u)-\bar{\phi}_i'(u)\right]-\frac{\lambda_i}{b_i}\sum_j p_{ij}\bar{\phi}_j(u), i\in I. \end{aligned} \tag{3.157}$$

Consequently, the functions $\bar{\phi}_i, i=1,...,m$ satisfy the following differential system of order 2:

$$\bar{\phi}_i''(u) = (\lambda_i - \frac{1}{b_j})\bar{\phi}_i'(u) + \frac{\lambda_i}{b_i}\bar{\phi}_i(u) - \frac{\lambda_i}{b_i}\sum_j p_{ij}\bar{\phi}_j(u), i \in I. \tag{3.158}$$

For $m=1$, we recover the differential equation

$$\phi''(u) = \left(\lambda - \frac{1}{b}\right)\phi'(u) \tag{3.159}$$

discussed in Gerber (1979).
As initial conditions, we have from (3.156)

$$\begin{aligned} &\lim_{u\to\infty}\bar{\phi}_i(u) = 1, \\ &\bar{\phi}_i'(0) = \lambda_i\bar{\phi}_i(0), i = 1,\dots,m. \end{aligned} \tag{3.160}$$

The interest of result (3.158) lies in the possibility it provides of stating the general solution as a linear combination of negative exponential functions, as in (3.154), usually with constant coefficients, which is always interesting from a numerical point of view.

Remark 3.6 Concerning the numerical point of view, Reinhard and Snoussi (2002) give another approach using a discrete time scale and recursive algorithms to obtain the joint distribution of the surplus prior to ruin and to compute the severity of ruin.

Nevertheless, let us mention that a prudent attitude of the insurance companies implies that the use of the non-ruin probability on an infinite time horizon is the best criteria and of course does not need to use the severity of ruin.

Chapter 8

RELIABILITY AND CREDIT RISK MODELS

In this chapter, the reader will first find a short summary of the basic notions of reliability and then the semi-Markov extensions.
After that, the classical problem of credit risk is also presented together with an analogy with reliability and it is shown how semi-Markov models are useful for this important topic of finance in connection with the new rules of the Basel Committee.

1 CLASSICAL RELIABILITY THEORY

Reliability theory is mainly concerned with the security of material fittings.
A first distinction must be made between simple and complex structures.
For a simple structure, it is possible to define what is called the lifetime of the considered system, defined as the r.v. T representing the time interval between time 0 and the time of the first failure, failure meaning that the system is out.
A complex system is composed of several simple components, from which failures have an impact, more or less important, on the way the system is working.

1.1 Basic Concepts

Let us consider a simple structure called the reliability system S having r.v. T as lifetime, T being defined on the probability space $(\Omega, \Im, P)$.

Definition 1.1 *The reliability function of S is given by the function U defined as*

$$U(t) = P(T > t), t \in [0, \infty). \tag{1.1}$$

$U(t)$ represents the probability that no failure happens before t. If F represents the distribution function of T, it is clear that for all non-negative t:

$$U(t) = 1 - F(t). \tag{1.2}$$

If the density function f of T exists, we obtain:

$$U(t) = \int_t^\infty f(u)du. \tag{1.3}$$

From now on, we always assume that the f or the derivative of U exists.

Definition 1.2 *The function r, defined as*

$$r(t)=\frac{f(t)}{1-F(t)}\left(=-\frac{U'(t)}{U(t)}\right),\ \ t>0, \tag{1.4}$$

is called the failure rate of the component.

Its meaning is simple: let us consider a time t such that the event $\{T>t\}$ occurs. From basic definitions of conditional probability (relation (6.4) of Chapter 1) and from relation (1.2), we can successively write:

$$\begin{aligned} P\left(T\le t+dt \middle| T>t\right) &= \frac{P\left(t<T\le t+dt\right)}{P(T>t)} \\ &= \frac{f(t)dt}{U(t)} \\ &= -\frac{U'(t)}{U(t)}dt \\ &= r(t)dt. \end{aligned} \tag{1.5}$$

Consequently, $r(t)dt$ simply represents the conditional probability of having a failure in the infinitesimal time interval $(t,t+dt)$ given that the component has no failure before or at time t. So, the value of the failure rate at time t is a risk measure to have suddenly a failure just after time t.
By integration, relation (1.4) gives:

$$U(t)=e^{-\int_0^t r(u)du} \tag{1.6}$$

provided that we suppose that $U(0)=1$.
From the last relation, it is clear that any non-negative function can be a failure rate if the following two conditions are satisfied:

- the function r is integrable on the positive half real line,
- $\int_0^\infty r(u)\,du=\infty.$ (1.7)

The mean lifetime $\bar{T}$ is just the mean for the d.f. F.
By integration by parts, it is possible to show that

$$\bar{T}=\int_0^\infty U(t)dt \tag{1.8}$$

and similarly if the variance σ^2 exists:

$$\sigma^2=2\int_0^\infty tU(t)dt \tag{1.9}$$

1.2 Classification Of Failure Rates

The first classification of failure rate types was given by Barlow and Proschan (1965) with the following definition.

Definition 1.3 *There are three types of failure rates:*

(i) *increasing failure rate type* (*in short IFR*) *iff*

$$\forall t_1, t_2 : t_1 < t_2 \Rightarrow r(t_1) < r(t_2)\,, \tag{1.10}$$

(ii) *decreasing failure rate type* (*in short DFR*) *iff*

$$\forall t_1, t_2 : t_1 < t_2 \Rightarrow r(t_1) > r(t_2)\,, \tag{1.11}$$

(iii) *constant failure rate type* (*in short CFR*) *iff*

$$\forall t_1, t_2 : t_1 < t_2 \Rightarrow r(t_1) = r(t_2)\,. \tag{1.12}$$

In the last case, let us write $r(t) = \lambda$; then relation (1.6) gives:

$$U(t) = e^{-\lambda t}, t \geq 0 \tag{1.13}$$

so that the d.f. of T is the negative exponential distribution of parameter λ (see Chapter 1, section 5.5).

Later Barlow and Prochan (1965) refine this classification with the following definition.

Definition 1.4 (i) A failure rate is of increasing failure rate average (in short IFRA) type (respectively of decreasing failure rate average (in short DFRA) type) iff the function $x \mapsto \frac{1}{x}\int_0^x r(t)dt, x \in [0,\infty)$ is increasing (respectively decreasing).

(ii) *A failure rate is of new better than used* (*in short NBU*) *type* (*respectively of old better than used* (*in short OBU*) *type*) *iff*

$$U(x+y) \leq (\geq) U(x)U(y), \forall x, y > 0. \tag{1.14}$$

The meaning of these last two types is simple; for the OBU type, for example, we can write inequality (1.13) in the form:

$$U(x) \geq \frac{U(x+y)}{U(y)} \tag{1.15}$$

or

$$P(T > x) \geq \frac{P(T > x+y)}{P(T > y)} \tag{1.16}$$

and finally

$$P(T > x) \geq P(T > x + y \,|\, T > y), \tag{1.17}$$

this last relation meaning that, given the event $\{T>y\}$, the conditional probability of the event $\{T>x+y\}$ is always smaller than the unconditional probability of the same event for y=0. In other terms, the fact of working up to time x always implies a wear phenomenon called *aging*.

Let us mention that it is possible to show (Barlow and Proschan (1965)) that the following inclusions are true:

$$IFR \subset IFRA \subset NBU,$$

$$DFR \subset DFRA \subset OBN.$$

Moreover, these inclusions are strict.
The general shape of a failure rate is the "bathtub" with three periods: in the beginning, it is of type DFR, then there is a time interval in which it is of exponential type and finally in a third and later period, of IFR type:

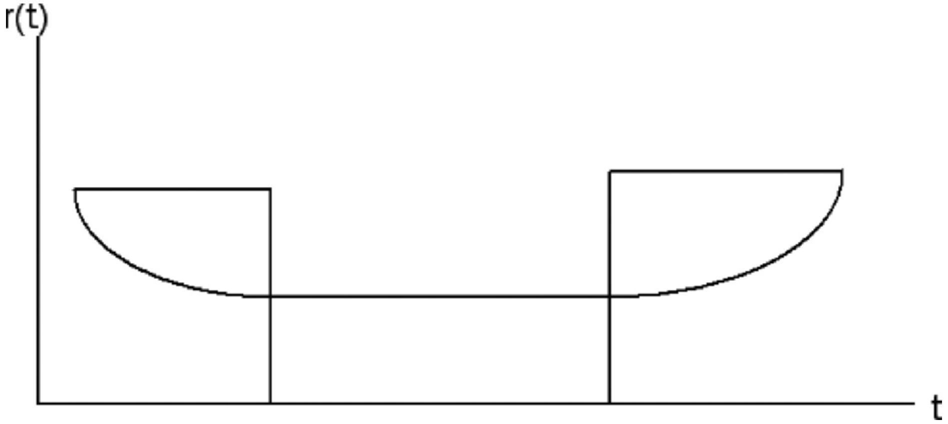

Figure 1.1: "Bathtub" shape of a failure rate

1.3 Main Distributions Used In Reliability

Referring to section 5 of Chapter 1, we will give the principal distributions used in reliability theory, together with the value of the failure rate and its type, if any.

(i) Poisson distribution of parameter λ;
(ii) Gamma distribution

$$\gamma(\lambda,r),\ r(t)=\frac{t^{r-1}e^{-\lambda t}}{\int_t^{\infty}u^{r-1}e^{-\lambda u}du}; \tag{1.18}$$

(iii) Weibull distribution of parameters λ,β,

$$r(t)=\lambda\beta\left(\lambda t\right)^{\beta-1}\ (\beta>1:IFR,\ \beta<1:DFR,\ \beta=1:EXP); \tag{1.19}$$

(iv) log-normal distribution of parameters μ,σ,

$$r(t)=\frac{e^{-(\ln t-\mu)^2/2\sigma^2}}{t\int_t^{\infty}\frac{e^{-(\ln u-\mu)^2/2\sigma^2}}{u}du}; \tag{1.20}$$

(v) the truncated normal law of parameter (μ,σ) for which the density is defined as

$$f(t)=\frac{1}{k\sigma\sqrt{2\pi}}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\left(=\frac{1}{k\sigma}\Phi'\left(\frac{t-\mu}{\sigma}\right)\right),$$
$$k=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{0}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt\left(=\Phi\left(-\frac{\mu}{\sigma}\right)\right), \tag{1.21}$$

Φ being defined in Chapter 1 as the distribution function of an $N(0,1)$ r.v.
The failure rate has the value

$$r(t)=\frac{e^{-\frac{(t-\mu)^2}{2\sigma^2}}}{\int_t^{\infty}e^{-\frac{(t-\mu)^2}{2\sigma^2}}du}\left(=\frac{\Phi'\left(\frac{t-\mu}{\sigma}\right)}{\sigma\Phi\left(\frac{t-\mu}{\sigma}\right)}\right). \tag{1.22}$$

1.4 Basic Indicators Of Reliability

Generally speaking, let us suppose now that the considered component is repairable, that is to say that if there is a failure at time *t*, we can repair the component using a random time *Y* and that after reparation, the component will start again with the same failure rate as before the reparation. This is the concept of minimal reparation.
The distribution function *G* of the repair time *Y* is called the maintainability function and the equivalent of the failure rate the repair rate function noted *s*, so that:

$$s(t)=\frac{G'(t)}{1-G(t)},$$
$$G(t)=1-e^{-\int_0^t s(u)du}. \tag{1.23}$$

The effect of considering possible reparations implies that we can now introduce a two-state system with as state space $\{0,1\}$, these two states representing respectively working and repair states.
Now the reliability system can have transition from one state to the other one: at time 0, the system is in the operating state or state 1 or up state; at the time of the first failure T_1, it goes into state 0 or down state for a time Y_1 and so on.
The time evolution of the system *S* is thus theoretically given by the sequence $\{T_1,Y_1,...,T_n,Y_n,...\}$ and from now on, let *Z*(*t*) represent the state of the considered system at time *t*.

Definition 1.5 *The basic indicators of reliability are:*
(i) *the mean time to failure (MTTF) :MTTF=E(T),*
(ii) *the mean time to repair (MTTR):MTTR=E(Y),*

(iii) *the point-wise* (*or instantaneous*) *availability:* $P(Z(t)=1)$,

(iv) *the steady-state availability:* $A=\lim_{t\to\infty} P(Z(t)=1)$,

(v) *the average availability on the interval* [*t,t+u*]:

$$\overline{A}(t,t+u)=\frac{1}{u}\int_t^{t+u} A(u)du, \tag{1.24}$$

(vi) *the limit average availability on the interval* [*t,t+u*]:

$$\overline{A}(t,t+u)=\lim_{t\to\infty}\frac{1}{t}\int_0^t A(u)du. \tag{1.25}$$

1.5 Complex and Coherent Structures

In general, a structure is composed of several simple components and the event failure of this complex structure depends on the way these components are working. Theoretically, this is given with the so-called structure function.
Let us suppose that all the *n* components of the complex structure *C* are simple, so that, for each *i*=1,...,*n*,

$$x_i=x_i(t)=\begin{cases}1, & \text{(operating or up state)},\\ 0, & \text{(failed or down state)}.\end{cases} \tag{1.26}$$

The state *x* of the structure *C* at time *t*, is given by

$$x(t)=\psi(x_1(t),\ldots,x_n(t))=\begin{cases}1, & \text{(up state)},\\ 0, & \text{(down state)},\end{cases} \tag{1.27}$$

where the function

$$\psi:\{0,1\}^n \mapsto \{0,1\} \tag{1.28}$$

is called the structure function of *C*.
Let us now give the following definitions.

Definition 1.6 *The complex component has a monotone structure iff*

$$\begin{aligned}&x_i\le y_i,\Rightarrow \psi(x_1,\ldots,x_n)\le\psi(y_1,\ldots,y_n),\ i=1,\ldots,n,\\ &\psi(0,\ldots,0)=0,\ \ \psi(1,\ldots,1)=1.\end{aligned} \tag{1.29}$$

Definition 1.7 *A component i* (*i*=1,...,*n*) *of a complex system is irrelevant iff the structure function is constant in* x_i*; otherwise, this component is called relevant.*

Definition 1.8 *The complex component has a coherent structure iff it is monotone and each component is relevant.*

Particular cases

(i) A *series structure* functions iff each component does.

(ii) A *parallel structure* functions if at least one component does.

(iii) A *k-out-of n structure* functions iff at least *k* of the *n* components function.

For these three types of structure, the structure functions are successively given by:

(i) $\psi\left(x_1,...,x_n\right)=\prod_{i=1}^{n} x_i \left(=\min_{i=1,...,n} x_i\right),$ (1.30)

(ii) $\psi\left(x_1,...,x_n\right)=\coprod_{i=1}^{n} x_i \left(=\max_{i=1,...,n} x_i\right),$ (1.31)

(iii) $\psi\left(x_1,...,x_n\right)=\begin{cases} 1, \sum_{i=1}^{n} x_i \geq k, \\ 0, \sum_{i=1}^{n} x_i < k. \end{cases}$ (1.32)

Remarks 1.1 (i) Barlow and Proschan (1965) have proved that for any coherent structure, we have the intuitive result

$$\prod_{i=1}^{n} x_i \leq \psi\left(x_1,...,x_n\right) \leq \coprod_{i=1}^{n} x_i. \tag{1.33}$$

(ii) If all the components are *independent* and if $U_i(i=1,...,n)$ is the reliability function of component *i*, then, we have for a *series* system:

$$U_\psi(t)=\prod_{i=1}^{n} U_i(t), \tag{1.34}$$

for a *parallel* system:

$$U_\psi(t)=1-\prod_{i=1}^{n}(1-U_i(t)), \tag{1.35}$$

and for a *k-out-of n* system:

$$U_\psi(t)=\sum_{i=k}^{n}\binom{n}{i}(U(t))^i(1-U(t))^{n-i} \text{ (for } U_i(t)=U(t), i=1,...,n). \tag{1.36}$$

Example 1.1 Let us consider a complex system for which component *i* (*i=1,…,n*) has a negative exponential distribution of parameter $\lambda_i(i=1,...,n)$.
If the structure is a series structure, it follows from result (1.34) that:

$$U_\psi(t)=\prod_{i=1}^{n} e^{-\lambda_i t}\left(=e^{-\left(\sum_{i=1}^{n}\lambda_i\right)t}\right) \tag{1.37}$$

and so, this structure has also a negative exponential distribution whose parameter $\lambda = \sum_{i=1}^{n} \lambda_i$.

Moreover as

$$\overline{T}_i = \frac{1}{\lambda_i}, i = 1, ..., n, \tag{1.38}$$

we have:

$$\frac{1}{\overline{T}} = \sum_{i=1}^{n} \frac{1}{\overline{T}_i}, \tag{1.39}$$

a result showing that MTTF for the complex structure is given by the harmonic mean of the MTTF of all the *n* components.

If all the components have the same reliability function, this last result gives:

$$\overline{T} = \frac{\overline{T}_1}{n}. \tag{1.40}$$

Example 1.2 Let us consider the *redundant structure* formed by a complex structure composed of *n* identical components in parallel, all having a negative exponential reliability function of parameter λ. From result (1.35), we get:

$$U_\psi(t) = 1 - (1 - e^{-\lambda t})^n, \tag{1.41}$$

and so from relation (1.4):

$$r_\psi(t) = n\lambda e^{-\lambda t} \frac{(1 - e^{-\lambda t})^{n-1}}{1 - (1 - e^{-\lambda t})^n}, \tag{1.42}$$

proving that here the failure rate is time dependent.

It is not difficult to show that:

$$\begin{aligned} &n > 1 \Rightarrow r_\psi(t) < \lambda, \\ &t \to 0 \Rightarrow r_\psi(t) \sim n\lambda(\lambda t)^{n-1}, \\ &t \to \infty \Rightarrow r_\psi(t) \sim \lambda \frac{1 - ne^{-\lambda t}}{1 - e^{-\lambda t}} \to \lambda. \end{aligned} \tag{1.43}$$

These last results show the effect of *redundancy*, which adds to a simple negative exponential component, $n-1$ supplementary components in parallel to improve the reliability.

This effect is important at the beginning and of course is time decreasing with time *t* to converge to λ.

For the MTTF, we have that:

$$\overline{T}_\psi = \overline{T} \sum_{k=1}^{n} \frac{1}{k}, \tag{1.44}$$

and so the ratios of the MTTF are given for example by :

N	1	2	3	4
MTTF ratio	1	1.5	1.83	2.08

Table 1.1: example of MTTF

This clearly shows that the effect of redundancy is not proportional to the number of added components.

2. STOCHASTIC MODELLING IN RELIABILITY THEORY

2.1 Maintenance Systems

In the last subsection, result (1.34) shows that for a series structure, the reliability function is highly decreasing with the number of components.

However, if the components are repairable, it is possible to interrupt the system momentarily during the reparation of the failed component and then to reinsert the component in the system and so on.

For such a possibility, one can construct a stochastic model (Mohan et al (1962)) to compute the main indicators given in section 1.4.

Let us assume that all the n components are independent with negative exponential distributions, respectively with parameters $\lambda_1,...,\lambda_n$, and that the repair time for component i (i=1,...,n) has a negative exponential distribution of parameter μ_i. All the repair times are also independent and of other and of on the working times of the n components.

Moreover there is no time loss to replace the repaired components in the system.

The evolution of the system can be seen as a successive sequence of working and repair times.

For example for n=1, the random sequence

$$\left(X_1,Y_1,X_2,Y_2,...,X_n,Y_n,...\right) \tag{2.1}$$

represents successively the working and repair times and if we introduce a two-state set {0,1}, so that, at time t, the system state $Z(t)$ is in state 1 if it is operating and in state 0 if it is under repair, then the process (2.1) is a continuous Markov process where the transition matrix of the imbedded Markov chain is given by:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.2}$$

and for which the conditional sojourn times are given by:

$$1 \mapsto 0 : F_{10}(t) = \begin{cases} 0, t < 0, \\ 1 - e^{-\lambda t}, t \geq 0, \end{cases}$$
$$0 \mapsto 1 : F_{01}(t) = \begin{cases} 0, t < 0, \\ 1 - e^{-\mu t}, t \geq 0. \end{cases} \tag{2.3}$$

With the notation of Chapter 3, we have here:

$$b_{10} = \eta_1 = \frac{1}{\lambda_1}, b_{01} = \eta_0 = \frac{1}{\mu_1}, \tag{2.4}$$

and for the stationary distribution of the imbedded Markov chain:

$$\pi_0 = \pi_1 = \frac{1}{2}. \tag{2.5}$$

We are interested in the following two transition probabilities:

$$\phi_{10}(t) = P(Z(t) = 0 \mid Z(0) = 1),$$
$$\phi_{01}(t) = P(Z(t) = 1 \mid Z(0) = 0). \tag{2.6}$$

They are given by the system (10.3) of Chapter 4.
For n=2, using Laplace transforms, it is possible to show that

$$\phi_{10}(t) = \frac{\lambda}{\lambda + \mu}(1 - e^{-(\lambda+\mu)t}),$$
$$\phi_{01}(t) = \frac{\mu}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)t}. \tag{2.7}$$

The asymptotic behaviour is given by relation (10.8) of Chapter 3, or here directly from result (2.7),

$$\lim_{t \to \infty} \phi_{10}(t) = \frac{\lambda}{\lambda + \mu},$$
$$\lim_{t \to \infty} \phi_{01}(t) = \frac{\mu}{\lambda + \mu}. \tag{2.8}$$

As here, we have:

$$MTTF = \frac{1}{\lambda}, MTTR = \frac{1}{\mu}, \tag{2.9}$$

relations (2.8) take the form

$$\lim_{t \to \infty} \phi_{10}(t) = \frac{MTTR}{MTTF + MTTR},$$
$$\lim_{t \to \infty} \phi_{01}(t) = \frac{MTBF}{MTTF + MTTR}. \tag{2.10}$$

Remark 2.1 a) Mohan et al (1962) also solved the case of a series system with n components, independent with negative exponential distributions, respectively with parameters $\lambda_1,\ldots,\lambda_n$, and that the repair time for component *i* (*i*=1,…,*n*) has a negative exponential distribution of parameter μ.

In terms of semi-Markov modelling, this means that the state process $\left(Z(t),t\geq 0\right)$ has as state space the set $\{1,...,n,Op\}$ where state *i* (*i*=1,...,*n*), means that the system is in the failure state due to component *i*, and state *Op* that the system is operating at time *t*.

With the same semi-Markov approach, it is possible to show that:

$$\begin{aligned}
&\phi_i(t)=\frac{\lambda_i}{\lambda+\mu}(1-e^{-(\lambda+\mu)t}),i=1,...,n,\\
&\phi_{Op}(t)=\frac{\mu+\lambda e^{-(\lambda+\mu)t}}{\lambda+\mu},\\
&\phi_0(t)(=1-\phi_{Op}(t))=\frac{\lambda-\lambda e^{-(\lambda+\mu)t}}{\lambda+\mu},\\
&\lambda=\lambda_1+\cdots+\lambda_n.
\end{aligned}\tag{2.11}$$

And consequently, the asymptotic behaviour is given by the following relations:

$$\begin{aligned}
&\lim_{t\to\infty}\phi_i(t)=\frac{\lambda_i}{\lambda+\mu},i=1,...,n,\\
&\lim_{t\to\infty}\phi_{Op}(t)=\frac{\mu+\lambda e^{-(\lambda+\mu)t}}{\lambda+\mu},\\
&\phi_0(t)=\frac{\lambda}{\lambda+\mu}.
\end{aligned}\tag{2.12}$$

b) For the general stochastic model of Mohan et al (1962), there exists a simple form of the asymptotic behaviour given by the following expressions:

$$\begin{aligned}
&\lim_{t\to\infty}\phi_i(t)=\frac{\lambda_i}{\mu_i}\frac{1}{1+\sum_{i=1}^{n}\frac{\lambda_i}{\mu_i}},i=1,...,n,\\
&\lim_{t\to\infty}\phi_{Op}(t)=\frac{1}{1+\sum_{i=1}^{n}\frac{\lambda_i}{\mu_i}},\\
&\phi_0(t)=\frac{\sum_{i=1}^{n}\frac{\lambda_i}{\mu_i}}{1+\sum_{i=1}^{n}\frac{\lambda_i}{\mu_i}}.
\end{aligned}\tag{2.13}$$

2.2. The Semi-Markov Model For Maintenance Systems

Let us consider a complex system having as state space the set $I=\{1,...,m\}$ with m finite.
The set I is partitioned in two non-void subsets U and D where U is the set of all functioning or up states and D all the failed or down states and at any time t, the considered system is in one of these states and transitions are of course possible. Any duration in one of the states of U is an operating time and any duration in a state of D a non-operating time; a transition from a state of U to a state of D means that there is a breakdown and a transition from a state of D to a state of U may be seen as the end of a time of reparation.

The basic assumption is that the process $Z=\{Z(t),t\geq 0\}$ is a semi-Markov process with kernel $\mathbf{Q}$.

As in section 2.1, the minimal case is considered with the semi-Markov process with $I=\{0,1\}$ where state 0 means breakdown state and state 1 operating state.

As before the main reliability indicators are:

(i) The availability functions defined as:

$$\begin{aligned} A_i(t) &= P\left(Z(t)\in U \mid Z(0)=i\right), i\in I, \\ A_i(s,t) &= P\left(Z(t)\in U \mid Z(s)=i\right), i\in I, \end{aligned} \tag{2.14}$$

respectively in the homogeneous and the non-homogeneous case.
If as in Chapter 3, we define by ϕ_{ij} the transition probability functions for the Z-process, for both the cases, we get:

$$\begin{aligned} A_i(t) &= \sum_{j\in U}\phi_{ij}(t), i\in I, \\ A_i(s,t) &= \sum_{j\in U}\phi_{ij}(s,t), i\in I. \end{aligned} \tag{2.15}$$

If, in the homogeneous case, the process is ergodic, we can also define the asymptotic availability as:

$$A_i(\infty)=\lim_{t\to\infty}A_i(t)=\sum_{j\in U}\frac{\pi_j\eta_j}{\sum_{k\in I}\pi_k\eta_k}. \tag{2.16}$$

(ii) The reliability functions giving the probability that the system is always working on the time interval [0,t],

$$\begin{aligned} R_i(t) &= P(Z(t)\in U \mid Z(0)=i), i\in U, \\ R_i(s,t) &= P(Z(t)\in U \mid Z(s)=i), i\in U. \end{aligned} \tag{2.17}$$

To compute these probabilities, we will now work with another kernel $\mathbf{Q}^D$ for which all the states of the subset D are changed into absorbing states, meaning that:

$$\begin{aligned} p_{ij}^D &= \delta_{ij}, i \in D, j \in I, \\ p_{ij}^D(s) &= \delta_{ij}, i \in D, j \in I, \end{aligned} \tag{2.18}$$

respectively for the homogeneous and the non-homogeneous case.
Doing so, in the two cases, we get:

$$\begin{aligned} R_i(t) &= \sum_{j \in U} \phi_{ij}^D(t), i \in U, \\ R_i(s,t) &= \sum_{j \in U} \phi_{ij}^D(s,t), i \in U, \end{aligned} \tag{2.19}$$

where of course, in both the homogeneous and the non-homogeneous cases, the matrix $\mathbf{\Phi}^D$ gives the probabilities transition for the semi-Markov process of kernel $\mathbf{Q}^D$.

(iii) the maintainability functions giving the probability that the system is down at time 0 in the homogeneous case and at time s in the non-homogeneous case and that the system will leave the set D within the time t,

$$\begin{aligned} M(t) &= 1 - P\left(Z(u) \in D,\ \forall u \in (0,t]\right), \\ M(s,t) &= 1 - P\left(Z(u) \in D,\ \forall u \in (s,t]\right). \end{aligned} \tag{2.20}$$

To compute these probabilities, we will now work with another kernel $\mathbf{Q}^U$ for which all the states of the subset U are changed in absorbing states, meaning that:

$$\begin{aligned} p_{ij}^U &= \delta_{ij}, i \in U, j \in I, \\ p_{ij}^U(s) &= \delta_{ij}, i \in U, j \in I. \end{aligned} \tag{2.21}$$

Doing so, respectively, we get:

$$\begin{aligned} M_i(t) &= \sum_{j \in U} \phi_{ij}^U(t), i \in D, \\ M_i(s,t) &= \sum_{j \in U} \phi_{ij}^U(s,t), i \in D, \end{aligned} \tag{2.22}$$

where of course, the matrix $\mathbf{\Phi}^U$ gives the probabilities transition for the semi-Markov process of kernel $\mathbf{Q}^U$.

In conclusion, we see that the computing of the main indicators for the semi-Markov model is simple from our numerical results given before.

2.3 A Classical Example

In this part the example given in Barlow and Proschan (1965) page 120 will be developed. It is supposed that there are two machines (computers in the original example) working in parallel. The system is formed by nine states. With three of these states $D = \{5,7,9\}$ the system is down, in the other six the system can work.

The MC that describes the system is given in **Table 2.1**.

0	$e^{-2\lambda t_0}$	0	0	0	$\frac{1-e^{-2\lambda t_0}}{2}$	0	$\frac{1-e^{-2\lambda t_0}}{2}$	0
0	0	$e^{-\lambda\gamma}$	0	0	0	0	0	$1-e^{-\lambda\gamma}$
0	0	0	$e^{-2\lambda t_0}$	0	$\frac{1-e^{-2\lambda t_0}}{2}$	0	$\frac{1-e^{-2\lambda t_0}}{2}$	0
$e^{-\lambda\gamma}$	0	0	0	$1-e^{-\lambda\gamma}$	0	0	0	0
0	0	0	0	0	1	0	0	0
0	0	$\frac{\theta}{\lambda+\theta}$	0	0	0	$\frac{\lambda}{\lambda+\theta}$	0	0
0	0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$\frac{\theta}{\lambda+\theta}$	0	0	0	0	0	$\frac{\lambda}{\lambda+\theta}$	0	0
0	0	0	0	0	0	0	1	0

Table 2.1: embedded MC

The embedded MC for the homogeneous case is the same as in **Table 2.1** and with parameter values:

$$\frac{1}{\lambda} = 35\,; \quad \frac{1}{\theta} = \frac{1}{6}\,; \ \gamma = 1\,; \ t_0 = 24\,. \tag{2.23}$$

In the non-homogeneous case the parameters are functions of time; more precisely:

$$\frac{1}{\lambda} = 40,\ldots,30\,; \quad \frac{1}{\theta} = \frac{1}{8},\cdots,\frac{1}{4}\,; \quad \gamma = 0.4,\ldots,1.2\,; \quad t_0 = 30,\ldots,20 \tag{2.24}$$

where:

$\frac{1}{\lambda}$ is the mean time to failure. In the non-homogeneous case, $\frac{1}{\lambda}$ is a decreasing function of the time in the sense that it goes from 40 to 30 in 11 steps.

$\frac{1}{\theta}$ is the mean time to perform emergency repair. In the non-homogeneous case $\frac{1}{\theta}$ is an increasing function that goes from $\frac{1}{8}$ to $\frac{1}{4}$ in 11 steps.

γ is the time to perform preventive maintenance. In the non-homogeneous case it is an increasing function that goes from 0.4 to 0.12 in 11 steps.
t_0 is the scheduled preventive maintenance period. In the non-homogeneous case it is a decreasing function that goes from 30 to 20 in 11 steps.
The following tables report the availability, reliability and maintainability functions both in the homogeneous and the non-homogeneous cases.
Each column of the tables reports the starting state. The rows represent time. In the homogeneous case the rows give probabilities after one period, two periods and so on. In the non-homogeneous case there are many results and, in our opinion, it was too tedious to report all of them. So, the first element of each row gives the starting time and the evaluation time.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
1	1	0.9957	1	0.9951	0.0101	0.9997	0.0718	0.9994	0.0914
2	0.9998	0.9948	0.9998	0.9939	0.1226	0.9995	0.1511	0.9993	0.1397
3	0.9997	0.9944	0.9997	0.9944	0.2309	0.9991	0.2338	0.9993	0.2307
4	0.9996	0.9933	0.9995	0.9938	0.2968	0.9988	0.2778	0.9992	0.2684
5	0.9994	0.9913	0.9994	0.9925	0.349	0.9985	0.3444	0.999	0.2693
6	0.9992	0.9902	0.9992	0.9913	0.4404	0.9981	0.381	0.9989	0.4194
7	0.9989	0.9874	0.999	0.99	0.4454	0.9978	0.5317	0.9987	0.4301
8	0.9985	0.9882	0.9986	0.9856	0.5283	0.9974	0.6241	0.9981	0.5139
9	0.9982	0.9874	0.9982	0.9859	0.6324	0.9973	0.6653	0.9976	0.6183
10	0.9979	0.9844	0.9979	0.9855	0.6841	0.9972	0.7126	0.9972	0.706
11	0.9974	0.9856	0.9974	0.9853	0.791	0.997	0.8027	0.9968	0.8289

Table 2.2: Homogeneous Availability Function

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
0-1	1	0.9988	1	0.998	0.0219	0.9995	0.1001	0.9999	0.0366
0-4	0.9998	0.9948	0.9997	0.9934	0.285	0.9991	0.2389	0.9993	0.3555
0-8	0.9983	0.9929	0.9986	0.9932	0.6412	0.9981	0.6565	0.9983	0.7292
0-11	0.9964	0.9926	0.996	0.9925	0.9302	0.9963	0.9145	0.996	0.9063
1-2	1	0.9972	1	0.9984	0.074	0.9993	0.1534	0.9997	0.1514
1-5	0.9995	0.9975	0.9996	0.9948	0.2726	0.9988	0.3956	0.9987	0.4324
1-11	0.9962	0.9923	0.996	0.9925	0.8983	0.9968	0.8969	0.9962	0.899
3-4	1	0.9969	1	0.9972	0.2086	0.9997	0.1335	0.9998	0.0703
3-11	0.9964	0.9931	0.9961	0.9933	0.9085	0.9973	0.8972	0.9964	0.9269
4-8	0.9987	0.9935	0.9994	0.9943	0.4083	0.9982	0.4683	0.9982	0.5145
4-11	0.9962	0.9947	0.9959	0.9922	0.8662	0.9972	0.8927	0.9961	0.8842
5-6	1	0.9941	1	0.9988	0.105	0.9998	0.1782	0.9999	0.1118
5-11	0.9966	0.9945	0.9963	0.9943	0.8486	0.997	0.8788	0.9962	0.8917
6-7	1	0.9951	1	0.9959	0.2093	0.999	0.2725	0.9996	0.1379
6-11	0.9959	0.9963	0.9963	0.9911	0.8157	0.9972	0.8351	0.9962	0.8564
7-9	0.9994	0.9881	0.9995	0.9916	0.5369	0.9979	0.53	0.9977	0.2606
7-11	0.9958	0.9923	0.9964	0.9931	0.8029	0.9968	0.8215	0.9963	0.8427
9-11	0.9959	0.9879	0.9984	0.99	0.784	0.9976	0.7754	0.9973	0.7889
10-11	1	0.9829	1	0.9823	0.7812	0.9941	0.7688	0.9942	0.7542

Table 2.3: Non-Homogeneous Point-wise Availability Function

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
1	1	0.998	1	0.9983	0	0.9998	0	0.9998	0
2	1	0.995	1	0.9948	0	0.9995	0	0.9992	0
3	0.9999	0.9911	0.9999	0.9937	0	0.9992	0	0.999	0
4	0.9997	0.991	0.9998	0.9928	0	0.9988	0	0.9986	0
5	0.9995	0.9903	0.9997	0.9891	0	0.9982	0	0.9984	0
6	0.9992	0.989	0.9995	0.9885	0	0.9981	0	0.9983	0
7	0.999	0.9871	0.9993	0.9864	0	0.9976	0	0.9977	0
8	0.9986	0.9832	0.999	0.9823	0	0.9968	0	0.9971	0
9	0.9982	0.9812	0.9987	0.9807	0	0.9967	0	0.9967	0
10	0.9976	0.9776	0.9982	0.9781	0	0.9961	0	0.9961	0
11	0.997	0.9765	0.9977	0.9764	0	0.9955	0	0.9954	0

Table 2.4: Homogeneous Reliability Function

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
0-1	1	0.9977	1	0.9991	0	0.9995	0	0.9999	0
0-4	0.9997	0.9909	0.9998	0.9936	0	0.9984	0	0.9992	0
0-8	0.9976	0.988	0.9979	0.9874	0	0.9974	0	0.9973	0
0-11	0.9921	0.9802	0.9926	0.9807	0	0.9933	0	0.993	0
1-2	1	0.9976	1	0.9982	0	0.9997	0	0.9995	0
1-5	0.9997	0.9922	0.9997	0.9938	0	0.999	0	0.9989	0
1-11	0.9927	0.9797	0.9927	0.9805	0	0.9928	0	0.9919	0
3-4	1	0.9993	1	0.9979	0	0.9998	0	0.9996	0
3-11	0.9934	0.9806	0.9933	0.9806	0	0.9932	0	0.9934	0
4-8	0.9992	0.9919	0.9987	0.9898	0	0.998	0	0.998	0
4-11	0.9935	0.98	0.9935	0.9804	0	0.9939	0	0.9934	0
5-6	1	0.9972	1	0.9952	0	0.9987	0	0.9974	0
5-11	0.9939	0.981	0.9937	0.982	0	0.9929	0	0.994	0
6-7	1	0.9971	1	0.9938	0	0.9988	0	0.9992	0
6-11	0.9946	0.981	0.9941	0.9822	0	0.994	0	0.9936	0
7-9	0.9987	0.9921	0.9995	0.9924	0	0.9965	0	0.9987	0
7-11	0.9943	0.9815	0.9947	0.9823	0	0.9941	0	0.993	0
9-11	0.9971	0.9834	0.9979	0.9838	0	0.9947	0	0.9945	0
10-11	1	0.9821	1	0.9821	0	0.9944	0	0.9939	0

Table 2.5: Non-Homogeneous Reliability Function

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
1	1	1	1	1	0.1384	1	0.0405	1	0.06
2	1	1	1	1	0.2119	1	0.1071	1	0.2089
3	1	1	1	1	0.222	1	0.1983	1	0.3333
4	1	1	1	1	0.3097	1	0.2815	1	0.4281
5	1	1	1	1	0.4346	1	0.3702	1	0.4716
6	1	1	1	1	0.555	1	0.4738	1	0.4933
7	1	1	1	1	0.6623	1	0.5361	1	0.5148
8	1	1	1	1	0.6841	1	0.6028	1	0.6627
9	1	1	1	1	0.7577	1	0.6577	1	0.7199
10	1	1	1	1	0.8174	1	0.7178	1	0.8112
11	1	1	1	1	0.8345	1	0.789	1	0.8481

Table 2.6: Homogeneous Maintainability Function

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
0-1	1	1	1	1	0.1384	1	0.0405	1	0.06
0-4	1	1	1	1	0.2119	1	0.1071	1	0.2089
0-8	1	1	1	1	0.222	1	0.1983	1	0.3333
0-11	1	1	1	1	0.3097	1	0.2815	1	0.4281
1-2	1	1	1	1	0.4346	1	0.3702	1	0.4716
1-5	1	1	1	1	0.555	1	0.4738	1	0.4933
1-11	1	1	1	1	0.6623	1	0.5361	1	0.5148
3-4	1	1	1	1	0.6841	1	0.6028	1	0.6627
3-11	1	1	1	1	0.7577	1	0.6577	1	0.7199
4-8	1	1	1	1	0.8174	1	0.7178	1	0.8112
4-11	1	1	1	1	0.8345	1	0.789	1	0.8481
5-6	1	1	1	1	0.0439	1	0.1272	1	0.0148
5-11	1	1	1	1	0.3685	1	0.3928	1	0.3081
6-7	1	1	1	1	0.6952	1	0.7669	1	0.6859
6-11	1	1	1	1	0.9312	1	0.9556	1	0.9951
7-9	1	1	1	1	0.1341	1	0.1304	1	0.0183
7-11	1	1	1	1	0.3636	1	0.3752	1	0.3966
9-11	1	1	1	1	0.8902	1	0.9444	1	0.9189
10-11	1	1	1	1	0.0253	1	0.0735	1	0.0108

Table 2.7: Non-Homogeneous Maintainability Function

3 STOCHASTIC MODELLING FOR CREDIT RISK MANAGEMENT

3.1 The Problem Of Credit Risk

At the present time, the credit risk problem is one of the most important contemporary problems and has been developed in the financial literature both from theoretical and practical points of view. It consists in computing the default probability of a firm.

Banks and other financial intermediaries are the type of firms that are the most concerned with evaluation of credit risk. There is a very wide range of literature on credit risk models (see for example, Bluhm et al (2002), Crouhy et al (2000)).

In the 1990s, Markov models were introduced to study credit risk problems. Many important papers on these kinds of models were published (see Jarrow et al (1997), Nickell et al (2000), Israel et al (2001), Hu et al (2002)), mainly for solving the problem of evaluation of transition matrices. In the paper by Lando and Skodeberg (2002) some problems regarding the duration of the transition are explored, but only recently models in which the randomness of time in the state transitions have been constructed (see D'Amico et al (2004), (2005a), (2005b), Vasileiou and Vassiliou (2006)).

By means of the semi-Markov model, it is possible to generalise the Markov models introducing the randomness of time for transitions between the states. Furthermore we think that the credit risk problem can be seen as a reliability financial model for the firm under study for which we would like to compute the default probability (D'Amico et al (2004), (2005a), (2005b)).

3.2 Construction Of A Rating Using The Merton Model For The Firm

In this section, we will develop an elaboration of a rating model using the classical Merton model for the firm (1974) and used in Creditmetrics, initialised by J-P Morgan as a sequel of the Riskmetrics computer program dedicated to the VaR methods.

In the Merton model (1974), the value V of the firm is modelled with a Black and Scholes stochastic differential equation with trend μ and instantaneous volatility σ so that its value time at t is given by

$$V(t) = V_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}, \tag{3.1}$$

V_0 being the value of the firm at time 0 and $W = \left(W(t), t \in [0,T]\right)$ a standard Brownian motion defined on the filtered probability space $\left(\Omega, \Im, (\Im_t), P\right)$.

If V_{def} is the threshold beyond which the firm defaults, called the threshold default, the probability P_{def} that the company defaults before time t is given by:

$$\begin{aligned} P_{déf}\left(V_{déf}, t\right) &= P\left(V(t) < V_{déf}\right) \\ &= P\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t) < \ln \frac{V_{déf}}{V_0}\right) \\ &= P\left(W(t) < \frac{1}{\sigma}\left(\ln \frac{V_{déf}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right). \end{aligned} \tag{3.2}$$

As, for all positive t, $W(t)/\sqrt{t}$ has a normal distribution, we get:

$$P_{def}\left(V_{def}, t\right) = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln \frac{V_{def}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right). \tag{3.3}$$

So, if we fix the value of P_{def}, we can compute the corresponding value of V_{def} using the quantiles of the normal distribution.

Let us suppose that we fix the default probability P_{def} to the corresponding quantile Z_{CCC}.

This means that if Z is below or equal to Z_{CCC}, with Z defined by:

$$Z = \frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right), \tag{3.4}$$

we get:

$$P_{def}\left(V_{def}, t\right) = \Phi\left(Z_{CCC}\right),$$
$$Z_{CCC} = \frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{def}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right). \tag{3.5}$$

On the contrary, if the value of Z is larger than Z_{CCC} but before the quantile Z_B, the rating given to the firm is noted *CCC* and so on. So we obtain a scale of increasing thresholds represented by:

$$Z_{CCC} < Z_B < Z_{BB} < Z_{BBB} < Z_A < Z_{AA} < Z_{AAA}, \tag{3.6}$$

assigning a credit rating or grade to firms as an estimate of their creditworthiness. If Z represents the observed value of Z for the considered firm, the scale used here is the rating use by the famous credit rating agencies Standard & Poor's and Moody's given below:

Zobs value	**notation**
Zobs<ZCCC	default
ZCCC<Zobs<ZB	CCC
ZB<Zobs<ZBB	B
ZBB<Zobs<ZBBB	BB
ZBBB<Zobs<ZA	BBB
ZA<Zobs<ZAA	A
ZAA<Zobs<ZAAA	AA
ZAAA<Zobs	AAA

Table 3.1: rating agencies

It is clear that the credit ratings depend on time t and also on the selection of the probabilities P_{def}, $P(Z_{CCC})$, $P(Z_B)$, $P(Z_{BB})$, $P(Z_{BBB})$, $P(Z_A)$, $P(Z_{AA})$, $P(Z_{AAA})$ chosen by the credit rating agency.

We can also compute the following probabilities:

$$\begin{aligned}
&P_{déf} = P(Z_{obs} < Z_{CCC}),\\
&P_{CCC} = P(Z_{CCC} < Z_{obs} < Z_B),\\
&P_B = P(Z_B < Z_{obs} < Z_{BB}),\\
&P_{BB} = P(Z_{BB} < Z_{obs} < Z_{BBB}),\\
&P_{BBB} = P(Z_{BBB} < Z_{obs} < Z_A),\\
&P_A = P(Z_A < Z_{obs} < Z_{AA}),\\
&P_{AA} = P(Z_{AA} < Z_{obs} < Z_{AAA}),\\
&P_{AAA} = P(Z_{AAA} < Z_{obs}),
\end{aligned} \tag{3.7}$$

and so:

$$\begin{aligned}
&P_B = P_{déf} + P_{CCC,}\\
&P_{déf} + P_{CCC} + P_B + \cdots + P_{AA} + P_{AAA} = 1.
\end{aligned} \tag{3.8}$$

Using relation (3.3), we get:

$$\begin{aligned}
&P_{déf} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{CCC}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right),\\
&P_{déf} + P_{CCC} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_B}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right),\\
&P_{déf} + P_{CCC} + P_B = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{BB}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right),\\
&\ldots\\
&P_{déf} + P_{CCC} + P_B + P_{BB} + \cdots + P_{AA} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AAA}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right);
\end{aligned} \tag{3.9}$$

where $V_{CCC} < V_B < V_{BB} < V_{BBB} < V_A < V_{AA} < V_{AAA}$ are the firm values corresponding to the rating $Z_{CCC} < Z_B < Z_{BB} < Z_{BBB} < Z_A < Z_{AA} < Z_{AAA}$ and so:

$$P_{déf} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{CCC}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),$$

$$P_{CCC} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln V\frac{Z_B}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right)-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{CCC}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),$$

$$P_B = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{BB}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right)-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_B}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right), \tag{3.10}$$

...

$$P_{AA} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AAA}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right)-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AA}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),$$

$$P_{AAA} = 1-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AAA}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right).$$

All these relations show how the grades are time dependent, which is why we will now study the dynamics of ratings

3.3 Time Dynamic Evolution Of A Rating

3.3.1 Time Continuous Model

In continuous time, the rating process is nothing else than the stochastic process defined by relation (3.4),

$$Z = \{Z_t, 0 \le t \le T\} \tag{3.11}$$

where the r.v. Z_t represents the credit rating at time t given by:

$$P_{déf}(V_t,t) = \Phi(Z_t),$$

or (3.12)

$$Z_t = \frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_t}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right).$$

Here, the grade Z_t represents exactly the value inside one of the classes defined above and no longer only the class.

Using relation (3.1) to substitute the value of V_t in (3.12), we get:

$$Z_t = \frac{W(t)}{\sqrt{t}}, t > 0, \tag{3.13}$$

so that

$$P\left(Z_{t+\Delta t} \le j \middle| Z_t\right) = P\left(\frac{W(t+\Delta t)}{\sqrt{t+\Delta t}} \le j \middle| \frac{W(t)}{\sqrt{t}} = i\right), \Delta t > 0, i, j > Z_{CCC}. \tag{3.14}$$

As the standard Brownian process has stationary and independent increments (see Chapter 1, section 9, **Definition 9.1**), we also get:

$$P\left(\frac{W(t+\Delta t)}{\sqrt{t+\Delta t}} \leq j \middle| \frac{W(t)}{\sqrt{t}} = i\right)$$
$$= P\left(W(t+\Delta t) - W(t) \leq j\sqrt{t+\Delta t} - W(t) \middle| \frac{W(t)}{\sqrt{t}} = i\right), \tag{3.15}$$

or using relation (3.14):

$$P(Z_{t+\Delta t} \leq j | Z_t) = P\left(\frac{W(t+\Delta t) - W(t)}{\sqrt{\Delta t}} \leq \frac{j\sqrt{t+\Delta t} - i\sqrt{t}}{\sqrt{\Delta t}} | Z_t = i\right)$$
$$= \Phi\left(\frac{j\sqrt{t+\Delta t} - i\sqrt{t}}{\sqrt{\Delta t}}\right), \tag{3.16}$$

the last equality coming from the normality of the increments of a standard Brownian motion.
We can also write this last result in the form:

$$P(Z_s \leq j | Z_t = i) = \Phi\left(\frac{j\sqrt{s} - i\sqrt{t}}{\sqrt{s-t}}\right). \tag{3.17}$$

The corresponding density function is given by:

$$\frac{d}{dj}\left(P(Z_s \leq j | Z_t = i)\right) = \frac{\sqrt{s}}{\sqrt{s-t}}\Phi'\left(\frac{j\sqrt{s} - i\sqrt{t}}{\sqrt{s-t}}\right). \tag{3.18}$$

This last result is correct only for $i \geq Z_{CCC}$. On the other hand, for $i < Z_{CCC}$, the default state being considered as an absorbing state, we have necessarily for $j \geq i$:

$$P(Z_s \leq j | Z_t = i) = 1. \tag{3.19}$$

In conclusion, as the transition probability given by (3.17) depends on both s and t and not only on $t-s$, we just prove that the Z process is *a non-homogeneous Markov process*, introduced in Chapter 3.

3.3.2 Discrete Continuous Model

Let us define $\{1,...,m\}$ as the set of the m credit ratings ranked in increasing order with the Moody scale: $1 = D_{def}$ (*default*),$2 = Z_{CCC}, ..., m = Z_{AAA}$.

Except for the extreme classes, the rating classes defined below will now be represented by their centres as follows:

$$\begin{array}{ll} (-\infty,1] : & 1 \\ (1,2] : & \frac{3}{2} \\ \dots & \\ (i-1,i] : & \frac{2i-1}{2} \\ \dots & \\ (m-1,m] : & \frac{2m-1}{2} \\ (m,\infty) : & m \end{array} \tag{3.20}$$

Let $Z_t = i$, i being a class centre different from 1; from result (3.17), we have that:

$$P(j-1 < Z_s \leq j | Z_t = i) = \Phi\left(\frac{j\sqrt{s}-i\sqrt{t}}{\sqrt{s-t}}\right) - \Phi\left(\frac{(j-1)\sqrt{s}-i\sqrt{t}}{\sqrt{s-t}}\right), \quad s > t. \tag{3.21}$$

To get a discrete time, let us suppose that we give notations at times $0, u, 2u, \dots, ku$ representing for example one year or a semester. Now transition probabilities become:

$$P(j-1 < Z_{ku+1} \leq j | Z_{ku} = i) = \Phi\left(\frac{j\sqrt{ku+1}-i\sqrt{ku}}{\sqrt{u}}\right) - \Phi\left(\frac{(j-1)\sqrt{ku+1}-i\sqrt{ku}}{\sqrt{u}}\right), k = 0,1,\dots . \tag{3.22}$$

Of course, if Z_{ku} equals Z_{Def}, we know from relation (3.19) that

$$P(j-1 < Z_{ku+1} \leq j | Z_{ku} = Z_D) = \begin{cases} 0, j > 1, \\ 1, j \leq 1. \end{cases} \tag{3.23}$$

Relations (3.21) and (3.22) define a sequence of probability transition matrices $\mathbf{P}(k)$, k=0,1,... with:

$$\mathbf{P}(k) = \left[p_{ij}(k) \right] \tag{3.24}$$

and

$$p_{ij}(k) = P(j-1 < Z_{ku+1} \leq j | Z_{ku} = i), i, j = 1,\dots,m, k = 0,1,\dots . \tag{3.25}$$

It follows that the credit rating process Z in discrete time $Z=(Z_{ku}, k=0,1,\dots)$ is what we call a non-homogeneous Markov chain defined in Chapter 3.

Of course, in the very particular and unrealistic case where the probability transition matrices $\mathbf{P}(k)$, k=0,1,... are independent of t, the process in discrete time $Z=(Z_{ku}, k=0,1,\dots,)$ is then a homogeneous Markov chain as defined in Chapter 2.

3.3.3 Example

In real economic life, credit rating agencies play a crucial role; they compile data on individual companies or countries to estimate their probability of default, represented by their scale of credit ratings at a given time and also by the probability of transitions for successive credit ratings.
A change in the rating is called a *migration*.
Migration to a higher rating will of course increase the value of a company's bond and decrease its yield, giving what we call a negative *spread,* as it has a lower probability of default, and the inverse is true with a migration towards a lower grade with consequently a positive spread.
Here we have an example of a possible transition matrix for migration from one year to the successive one:

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**	**Total**
AAA	0.90829	0.08272	0.00736	0.00065	0.00066	0.00014	0.00006	0.00012	1
AA	0.00665	0.9089	0.07692	0.00583	0.00064	0.00066	0.00029	0.00011	1
A	0.00092	0.0242	0.91305	0.05228	0.00678	0.00227	0.00009	0.00041	1
BBB	0.00042	0.0032	0.05878	0.87459	0.04964	0.01078	0.0011	0.00149	1
BB	0.00039	0.00126	0.00644	0.0771	0.81159	0.08397	0.0097	0.00955	1
B	0.00044	0.00211	0.00361	0.00718	0.07961	0.80767	0.04992	0.04946	1
CCC	0.00127	0.00122	0.00423	0.01195	0.0269	0.11711	0.64479	0.19253	1
D	0	0	0	0	0	0	0	1	1

Table 3.2 : Example of transition matrix of credit ratings

We clearly see that the probabilities of no migration, given by the elements of the principal diagonal, are the highest elements of the matrix but that they decrease with the poor quality of the rating.
Here, we see for example that a company with rank AA has more or less nine chances out of ten to keep its rating next year, but it will move to rank AAA with only six chances in one thousand.
On the other hand, a company with a CCC as rating will be in default next year with twenty chances out of a hundred.
As a more real example, the next table gives the transition probability matrix of credit ratings of *Standard & Poor's* for year 1998 (see ratings performance, Standard & Poor's) for a sample of 4014 companies.
Let us point out the presence of a "new" state called N.R. (*rating withdrawn*) meaning that for a company in such a state, the rating has been withdrawn and that this event does not necessarily lead to default the following year, thus explaining the last row of the above matrix.

Effec.		AAA	AA	A	BBB	BB	B	CCC	D	N.R.	Total
165	**AAA**	90.3	6.1	0	0.61	0	0	0	0	3.03	100
560	**AA**	0.18	90	5.71	0.18	0	0	0	0	4.29	100
1095	**A**	0.09	1.5	87.22	5.11	0.18	0	0	0	5.94	100
896	**BBB**	0	0	2.79	84.93	4.46	0.67	0.22	0.34	6.59	100
619	**BB**	0.32	0.2	0.16	5.33	75.44	5.98	2.75	0.65	9.21	100
649	**B**	0	0	0.15	0.62	6.16	76.3	5.09	4.47	7.24	100
30	**CCC**	0	0	3.33	0	0	20	33.3	36.67	6.67	100
	N.R.	0	0	0	0	0	0	0	0	100	100
4014											

Table 3.3: example with rating withdrawn

Here, we see for example that companies in state AA will not be in default the next year but that 5.71 % of them will degrade to simple A and 18 % to a BBB and 0.18 will upgrade to an AAA.
Under the assumption of a homogeneous Markov chain, we obtain the following results:

(i) *the probability that an AA company defaults after two years:*
$P^{(2)}(D/AA)$=0.0018 $\cdot$ 0.0034=0.0006%,
which is still very low.
(ii) *the probability that a BBB company defaults in one of the next two years :*
This probability is given by:

$$P(D/BBB;2) = P(D/BBB) + P(BBB/BBB)P(D/BBB)$$
$$+P(BB/BBB)P(D/BB) + P(B/BBB)P(D/B) + P(CCC/BBB)P(D/CCC)$$
$$= 0.34\% + (84.93\% \cdot 0.34\%)$$
$$+(4.46\% \cdot 0.65\%)+(0.67\% \cdot 4.47\%)+(0.22\% \cdot 36.67\%)$$
$$=0.77\%.$$

(iii) *the probability for a company BBB to default between year 1 and year 2:*
Using the standard definition of conditional probability (see Chapter 1) we get
P(*D at* 2/*non-def. at* 1) = P(*D at* 2 & *non-def. at* 1)/ *P*(*non-def. at* 1)
=(0.77%-0.34%)/(1-0.34%)
=0.43%.

Let us point out that these illustrative results are true under the homogeneous Markov chain model and moreover give similar results for all the companies of the panel in the same credit rating.
In fact, in real life applications, credit rating agencies also study each company on its own account so that specific information is also determining for giving the final grade.

3.3.4 Rating And Spreads On Zero Bonds

Let us first recall that a zero-coupon bond is a contract paying a known fixed amount called the *principal*, at some given future date, called the *maturity date*. So if the principal is one monetary unit and T the maturity date, the value of this zero-coupon at time 0 is given by:

$$B(0,T) = e^{-\delta T} \tag{3.26}$$

if δ is the considered constant instantaneous intensity of interest rate.
Of course, the investor in zero-coupons must take into account the risk of default of the issuer. To do so, we consider that, in a risk neutral framework, the investor has no preference between the two following investments:

(i) to receive almost certainly at time 1 the amount e^{δ} as counterpart of the investment at time 0 of one monetary unit,
(ii) to receive at time 1 the amount $e^{(\delta+s)} (s>0)$ with probability $(1-p)$ or 0 with probability p, as counterpart of the investment at time 0 of one monetary unit, p being the default probability of the issuer.
The positive quantity s is called the *spread* with respect to the non-risky instantaneous interest rate δ as counterpart of this risky investment in zero-coupon bonds.
From the indifference given above, we obtain the following relation:

$$e^{\delta} = (1-p)e^{(\delta+s)} \tag{3.27}$$

or

$$1 = (1-p)e^{s}, \tag{3.28}$$

$$s = -\ln(1-p); \tag{3.29}$$

$$s \approx p,$$

$$s \cong p + \frac{1}{2}p^2. \tag{3.30}$$

Let us now consider a more positive and realistic situation in which the investor can get an amount $\alpha, (0<\alpha<1)$ if the issuer defaults at maturity or before.
In this case, the expectation equivalence principle relation (3.27) becomes:

$$e^{\delta} = (1-p)e^{\delta+s} + p\alpha e^{\delta}, \tag{3.31}$$

or

$$1 = (1-p)e^{s} + p\alpha. \tag{3.32}$$

It follows that in this case the value of the spread satisfies the equation

$$e^{s} = \frac{1-p\alpha}{1-p} \tag{3.33}$$

and so the spread value is

$$s = \ln \frac{1 - p\alpha}{1 - p}. \tag{3.34}$$

As above, using the Mac Laurin formula respectively of order 1 and 2, we obtain the two following approximations for the spread:

$$\begin{aligned} s &\approx \frac{p}{1-p}(1-\alpha), \\ s &\approx \frac{p}{1-p}(1-\alpha) - \frac{1}{2}\left(\frac{p}{1-p}(1-\alpha)\right)^2. \end{aligned} \tag{3.35}$$

4 CREDIT RISK AS A RELIABILITY MODEL

4.1 The Semi-Markov Reliability Credit Risk Model

As we already know, the credit risk problem can be seen as a reliability problem in which the rating process, carried out by the rating agency, gives a reliability degree of a firm bond and moreover, the default state can be seen as a down state and an absorbing state.

From relations (2.15) and (2.19) it results that in this case the concept of reliability and availability coincide.

We know that rating agencies like Standard & Poor's, Moody's or Fitch give each examined firm a rating. In the preceding subsections, we used the S&P simplified model giving eight kinds of ratings:

AAA, AA, A, BBB, BB, B, CCC, D,

where the states are in decreasing order depending on the "reliability" of their debts, and the state D means default (for the precise definition of each state see Crouhy et al (2001)).

In order to apply reliability models in a credit risk environment it is possible to consider, following S&P classification, the first seven states as "good" states and the D state as the only bad state and apply our semi-Markov reliability models to the credit risk problem.

The state D will be an absorbing state, because once the state is reached, in the sense that the firm is not in position to pay its debts and so therefore defaults, it is not possible to exit from the state.

Furthermore in this case we are interested only in the $R(t)$ function; the $A(t)$ and $M(t)$ functions are meaningless in this environment.

$R_i(t)$ gives the probability that the system was always working up to the time t given that the system was in the working state i at time 0.

In this case the reliability model is substantially simplified and to get all the results that are relevant in the credit risk case, it suffices to solve the semi-Markov evolution equation only once to get the following probabilities:

1) $\phi_{ij}(t)$ and $\phi_{ij}(s,t)$ representing respectively the probabilities to be in the state j after a time t starting in the state i at time 0 in the homogeneous case and starting at time s in the state i in the non-homogeneous case. The semi-Markov environment takes into account the different probabilities of state changes during the permanence of the system in the same state (duration problem);

2) $R_i(t) = \sum_{j \in U} \phi_{ij}(t)$ and $R_i(s,t) = \sum_{j \in U} \phi_{ij}(s,t)$, representing respectively the probabilities that the system never goes into the default state in a time t in the homogeneous case and from time s to time t in the non-homogeneous one;

3) $1 - H_i(t)$ and $1 - H_i(s,t)$, representing the probabilities that in a time interval t, in the homogeneous case, and from time s to time t, in the non-homogeneous case, no one new rating evaluation was done for the firm;

4) $\varphi_{ij}(t)$ and $\varphi_{ij}(s,t)$ representing the probabilities to get the rank j at the next rating if the previous state was i and not one rating evaluation was made up to the time t in the homogeneous case and from time s to time t in the non-homogeneous one. In this way, for example, if the transition to the default state is possible and if the system doesn't move for a time t from the state i, we know the probability that in the next transition the system will go to the default state.

They are defined by the following relations:

$$\varphi_{ij}(t) = \frac{p_{ij} - Q_{ij}(t)}{1 - H_i(t)}, \tag{4.1}$$

$$\varphi_{ij}(s,t) = \frac{p_{ij}(s) - Q_{ij}(s,t)}{1 - H_i(s,t)}. \tag{4.2}$$

4.2. A Homogeneous Case Example

Now we give an example using the transition matrix given in Jarrow et al (1997). This example will be chosen in the homogeneous case.

This matrix was constructed starting from the one year transition matrix given in Standard & Poor's Credit Review (1993).

We report the matrix for the sake of completeness.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.891	0.0963	0.0078	0.0019	0.003	0	0	0
AA	0.0086	0.901	0.0747	0.0099	0.0029	0.0029	0	0
A	0.0009	0.0291	0.8896	0.0649	0.0101	0.0045	0	0.0009
BBB	0.0006	0.0043	0.0656	0.8428	0.0644	0.016	0.0018	0.0045
BB	0.0004	0.0022	0.0079	0.0719	0.7765	0.1043	0.0127	0.0241
B	0	0.0019	0.0031	0.0066	0.0517	0.8247	0.0435	0.0685
CCC	0	0	0.0116	0.0116	0.0203	0.0754	0.6492	0.2319
D	0	0	0	0	0	0	0	1

Table 4.1: 1 year transition matrix

The matrix value for d.f. are not known and so we construct these d.f. by means of random number generators.
We report the results at time 5 and at time 10 of the matrix $\phi_{ij}(t)$ respectively in **Table 4.2.1** and **Table 4.2.2**.
For example the element 0.04326 that is in row **AA** and in column **A** represents the probability that a firm that at time 0 has a rating **AA** will have rating **A** at time 5.

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.93129	0.06044	0.00504	0.00148	0.00164	0.00009	0.00000	0.00001
AA	0.00464	0.94420	0.04326	0.00519	0.00100	0.00165	0.00002	0.00005
A	0.00051	0.01505	0.94403	0.02950	0.00697	0.00330	0.00004	0.00060
BBB	0.00030	0.00295	0.03704	0.90384	0.04110	0.00976	0.00105	0.00397
BB	0.00023	0.00148	0.00572	0.04727	0.85624	0.05887	0.00908	0.02111
B	0.00000	0.00096	0.00195	0.00351	0.03377	0.89002	0.02404	0.04575
CCC	0.00000	0.00004	0.00474	0.00535	0.01258	0.03479	0.85292	0.08958
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	1.00000

Table 4.2.1: probabilities $\phi_{ij}(5)$

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.83968	0.13696	0.01488	0.00375	0.00415	0.00047	0.00003	0.00008
AA	0.01084	0.86440	0.10055	0.01526	0.00433	0.00421	0.00012	0.00030
A	0.00141	0.03991	0.84668	0.08517	0.01638	0.00807	0.00032	0.00206
BBB	0.00086	0.00749	0.08702	0.78071	0.08579	0.02549	0.00327	0.00937
BB	0.00056	0.00344	0.01366	0.09229	0.69959	0.13097	0.01814	0.04135
B	0.00003	0.00279	0.00512	0.01162	0.06732	0.75319	0.05575	0.10419
CCC	0.00001	0.00029	0.01329	0.01436	0.02313	0.07803	0.61935	0.25154
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	1.00000

Table 4.2.2: probabilities $\phi_{ij}(10)$

Table 4.3 gives the reliability results $R_i(t)$, probabilities to have no default in a time *t* (row index) starting in the state *i* (column) at time 0.

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
1	1.00000	1.00000	0.99987	0.99933	0.99846	0.99642	0.99294	0.0
2	1.00000	1.00000	0.99975	0.99884	0.99461	0.98808	0.98146	0.0
3	1.00000	0.99999	0.99969	0.99789	0.98908	0.97527	0.96374	0.0
4	0.99999	0.99997	0.99961	0.99715	0.98624	0.97029	0.94233	0.0
5	0.99999	0.99995	0.99940	0.99603	0.97889	0.95425	0.91042	0.0
6	0.99998	0.99992	0.99917	0.99505	0.97436	0.94749	0.89800	0.0
7	0.99997	0.99989	0.99888	0.99418	0.97144	0.93795	0.84898	0.0
8	0.99995	0.99984	0.99856	0.99334	0.96771	0.92535	0.79778	0.0
9	0.99994	0.99978	0.99825	0.99210	0.96446	0.90689	0.77184	0.0
10	0.99992	0.99970	0.99794	0.99063	0.95865	0.89581	0.74846	0.0

Table 4.3: probabilities of not having a default

Table 4.4 gives probabilities $1-H_i(t)$ to remain always in the starting state without transitions.

	AAA	AA	A	BBB	BB	B	CCC	D
1	0.98490	0.89746	0.86572	0.92634	0.86317	0.86674	0.94774	1.0
2	0.82635	0.82919	0.74506	0.77373	0.78411	0.75612	0.92103	1.0
3	0.74275	0.75242	0.68724	0.65713	0.64732	0.66181	0.87814	1.0
4	0.57977	0.73210	0.55915	0.59711	0.60133	0.58323	0.79454	1.0
5	0.47763	0.51794	0.47518	0.45947	0.47929	0.43098	0.65872	1.0
6	0.37730	0.41739	0.35444	0.36779	0.42974	0.30765	0.56190	1.0
7	0.30913	0.32773	0.26773	0.29968	0.30514	0.24540	0.41717	1.0
8	0.23808	0.25246	0.22929	0.17914	0.27208	0.15297	0.25461	1.0
9	0.11174	0.21338	0.12389	0.14214	0.13721	0.11744	0.15293	1.0
10	0.08543	0.02793	0.06785	0.05651	0.04622	0.07177	0.05478	1.0

Table 4.4: probabilities to remain in the starting state

Lastly, **Tables 4.5.1** and **4.5.2** give probabilities $\varphi_{ij}(t)$ at 5 years and 10 years. For example 0.07128 represents the probability that a firm that was at time 0 in state **AA** and remained in this state up to time 5 will then have the next transition in state **A**.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.89611	0.09045	0.00858	0.00157	0.00329	0.00000	0.00000	0.00000
AA	0.00861	0.90209	0.07128	0.01090	0.00417	0.00296	0.00000	0.00000
A	0.00102	0.03494	0.86517	0.08440	0.00935	0.00405	0.00000	0.00107
BBB	0.00078	0.00436	0.06867	0.83756	0.06478	0.01815	0.00234	0.00336
BB	0.00041	0.00223	0.00690	0.06408	0.78794	0.11412	0.01027	0.01405
B	0.00000	0.00264	0.00357	0.00866	0.05087	0.81139	0.05240	0.07046
CCC	0.00000	0.00000	0.01073	0.01009	0.01212	0.06289	0.68286	0.22131
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	1.00000

Table 4.5.1: probabilities $\varphi_{ij}(5)$

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.98240	0.00781	0.00808	0.00063	0.00108	0.00000	0.00000	0.00000
AA	0.03042	0.83757	0.08618	0.02700	0.01034	0.00849	0.00000	0.00000
A	0.00107	0.04218	0.85463	0.09107	0.00919	0.00071	0.00000	0.00115
BBB	0.00072	0.00028	0.04913	0.87719	0.04199	0.02433	0.00251	0.00386
BB	0.00033	0.00285	0.00795	0.11255	0.64711	0.17648	0.01645	0.03627
B	0.00000	0.00211	0.00409	0.00684	0.06930	0.79561	0.03869	0.08336
CCC	0.00000	0.00000	0.00456	0.00604	0.03402	0.10958	0.50718	0.33864
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	1.00000

Table 4.5.2: probabilities $\varphi_{ij}(10)$

4.3 A Non-Homogeneous Case Example

Now we give a non-homogeneous example using as basis the transition matrices given in Table 15 of Standard & Poor's (2001).
In these matrices the state No Rating was present. Starting from the data reported in the S&P publication, the non-homogeneous transition matrix was constructed. Each element $p_{ij}(s)$ of the embedded non-homogeneous Markov chain should be constructed directly from the data.
Constructing the MC, all possible transitions from state *i* to state *j* starting from the year *s* should be taken into account. But we do not have the raw data and so we use the one year transition matrices given in the Standard & Poor's publication.
The publication reports 20 years of history (one year transition matrices from 1981 up 2000). The example covers from time 0, corresponding to the year 1981, to time 19, corresponding to the year 2000.
Table 4.6 reports three years of the non homogeneous embedded MC.

TRANSITION MATRICES								
	MATRIX AT TIME 0							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.92450	0.07550	0	0	0	0	0	0
AA	0.01990	0.91045	0.06965	0	0	0	0	0
A	0	0.04760	0.88406	0.06624	0.00210	0	0	0
BBB	0	0	0.04870	0.90260	0.04870	0	0	0
BB	0	0	0.00924	0.04631	0.62960	0.31023	0.00462	0
B	0	0	0.01240	0	0.04940	0.91351	0.02470	0
CCC	0	0	0	0	0	0.09090	0.90910	0
D	0	0	0	0	0	0	0	1.00000
	MATRIX AT TIME10							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.97070	0.02930	0	0	0	0	0	0
AA	0.00485	0.88460	0.11055	0	0	0	0	0
A	0	0.02129	0.88672	0.07783	0.01240	0.00176	0	0
BBB	0	0	0.04247	0.89077	0.05151	0.00915	0	0.00610
BB	0	0	0.00397	0.06742	0.74600	0.10714	0.03575	0.03972
B	0	0.00975	0.00321	0.00654	0.03912	0.78181	0.05862	0.10095
CCC	0.02269	0	0	0	0.02269	0.04549	0.56819	0.34093
D	0	0	0	0	0	0	0	1.00000
	MATRIX AT TIME19							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.96115	0.02776	0.01108	0	0	0	0	0
AA	0.00986	0.87568	0.11113	0.00332	0	0	0	0
A	0	0.02493	0.90134	0.06779	0.00428	0.00083	0	0.00083
BBB	0	0.00178	0.02259	0.92584	0.03703	0.00638	0.00272	0.00366
BB	0	0	0.00362	0.04074	0.87290	0.05871	0.01202	0.01202

B	0	0	0.00338	0.00338	0.03738	0.83230	0.04533	0.07824
CCC	0	0	0	0	0.01296	0.06489	0.59745	0.32470
D	0	0	0	0	0	0	0	1.00000

Table 4.6: Embedded NHMC

To apply the model it is necessary to construct also the d.f. of the waiting times in each state *i,* given that the state successively occupied is known. As we do not have these data either, we construct them by means of random number generators.

Probabilities $H_i(s,t)$ to remain in the state from *s* to *t* without any transition are reported in **Table 4.7.** For example the element 0.55706 represents the probability that the rating **AA** has no other rating evaluation from the time 0 up to the time 9.

Probability no movement								
TIMES		**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
0	1	0.90856	0.96379	0.98697	0.93658	0.93951	0.92026	0.94220
0	2	0.84173	0.92113	0.94230	0.88791	0.90524	0.86158	0.90722
0	3	0.78727	0.88494	0.88535	0.84758	0.85096	0.77361	0.89632
0	8	0.58795	0.59798	0.69578	0.65079	0.66009	0.62475	0.69739
0	9	0.51816	0.55706	0.68355	0.57614	0.58619	0.60454	0.64461
0	10	0.51338	0.50773	0.59941	0.52915	0.50759	0.59869	0.54138
0	17	0.13144	0.10865	0.14796	0.16612	0.16478	0.15852	0.13142
0	18	0.09384	0.09523	0.09996	0.09903	0.09947	0.10168	0.10572
0	19	0.02224	0.03585	0.03916	0.04847	0.07476	0.07823	0.06299
1	2	0.91830	0.94635	0.93876	0.96978	0.99286	0.90339	0.94856
1	11	0.42990	0.46490	0.48222	0.55546	0.44814	0.49398	0.45959
1	19	0.00536	0.07610	0.06425	0.02936	0.08739	0.06138	0.05097
2	3	0.94570	0.96836	0.90135	0.93726	0.98392	0.98624	0.97455
2	11	0.53603	0.44457	0.41834	0.42495	0.55929	0.52844	0.47755
2	19	0.09757	0.05980	0.07554	0.07681	0.06381	0.07286	0.05074
5	6	0.96537	0.92527	0.93849	0.97609	0.85933	0.97116	0.94845
5	13	0.55308	0.42610	0.43405	0.45723	0.38602	0.47904	0.47031
5	19	0.06883	0.02407	0.01707	0.01610	0.03967	0.03351	0.02644
7	8	0.94296	0.90031	0.90565	0.86546	0.86446	0.90422	0.88623
7	14	0.35994	0.44772	0.40121	0.33883	0.37718	0.42686	0.53458
7	19	0.05654	0.04441	0.02632	0.05958	0.01941	0.07475	0.07226
10	11	0.88047	0.94281	0.82722	0.88183	0.79898	0.88194	0.86912
10	15	0.40490	0.53878	0.39667	0.45283	0.32490	0.50327	0.41918
10	19	0.06716	0.00235	0.09389	0.06345	0.02003	0.08191	0.03364
13	14	0.96214	0.73636	0.90271	0.69666	0.65322	0.86464	0.88569
13	17	0.52826	0.27128	0.46206	0.30303	0.17601	0.45055	0.48379
13	19	0.00934	0.08777	0.03932	0.06692	0.08494	0.02828	0.02014
17	18	0.59856	0.51260	0.73049	0.30187	0.27078	0.73709	0.32717
17	19	0.04785	0.06337	0.00726	0.06841	0.07776	0.01435	0.05934

Table 4.7: probabilities to remain in the starting state without transitions

Tables 4.8.1 and **4.8.2** give the probabilities $\varphi_{ij}(s,t)$ that the next transition from the state i will be to the state j given that there is no transition from the time s to the time t.

$\varphi_{ij}(s,t)$ Prob. Next State Without Transitions from s to t								
	TIME 0-1							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.92523	0.07477	0	0	0	0	0	0
AA	0.02061	0.90773	0.07166	0	0	0	0	0
A	0	0.04379	0.88959	0.06456	0.00206	0	0	0
BBB	0	0	0.05006	0.90070	0.04924	0	0	0
BB	0	0	0.00926	0.04740	0.63802	0.30051	0.00481	0
B	0	0	0.01327	0	0.05037	0.91098	0.02537	0
CCC	0	0	0	0	0	0.09064	0.90936	0
D	0	0	0	0	0	0	0	1.00000
	TIME 0-10							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.92792	0.07208	0	0	0	0	0	0
AA	0.01978	0.90729	0.07293	0	0	0	0	0
A	0	0.04515	0.89622	0.05644	0.00219	0	0	0
BBB	0	0	0.04520	0.91235	0.04245	0	0	0
BB	0	0	0.00933	0.04906	0.63862	0.29823	0.00476	0
B	0	0	0.01053	0	0.04094	0.92444	0.02410	0
CCC	0	0	0	0	0	0.07973	0.92027	0
D	0	0	0	0	0	0	0	1.00000
	TIME 0-19							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.73240	0.26760	0	0	0	0	0	0
AA	0.03252	0.83111	0.13637	0	0	0	0	0
A	0	0.11417	0.85701	0.02554	0.00328	0	0	0
BBB	0	0	0.08342	0.83961	0.07698	0	0	0
BB	0	0	0.00180	0.03527	0.77843	0.18296	0.00154	0
B	0	0	0.01377	0	0.06237	0.91291	0.01095	0
CCC	0	0	0	0	0	0.03438	0.96562	0
D	0	0	0	0	0	0	0	1.00000

Table 4.8.1: probabilities $\varphi_{ij}(0,t)$

For example the element 0.07293 gives the probability that the next transition from the rating **AA** will be to the rating **A**, given that from the time 0 up to the time 10 there will be no *real* or *virtual* transitions; by virtual transition we denote the fact that the next transition is in the same state.

$\varphi_{ij}(s,t)$ Prob. Next State Without Transitions from s to t								
	TIME 15-16							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.94075	0.05343	0.00582	0	0	0	0	0

AA	0.00410	0.93332	0.06258	0	0	0	0	0
A	0	0.02803	0.95461	0.01667	0.00070	0	0	0
BBB	0.00156	0	0.07437	0.90203	0.02028	0.00175	0	0
BB	0	0	0.00782	0.06731	0.86642	0.04816	0.00506	0.00522
B	0	0	0.00266	0.00548	0.09608	0.84881	0.01684	0.03012
CCC	0	0	0	0	0.05549	0.11808	0.77923	0.04720
D	0	0	0	0	0	0	0	1.00000
	TIME 15-17							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.95837	0.03828	0.00335	0	0	0	0	0
AA	0.00763	0.87026	0.12211	0	0	0	0	0
A	0	0.02601	0.95488	0.01829	0.00082	0	0	0
BBB	0.00217	0	0.07649	0.89866	0.02092	0.00176	0	0
BB	0	0	0.00836	0.10192	0.84363	0.03558	0.00396	0.00655
B	0	0	0.00237	0.00557	0.08279	0.87605	0.01026	0.02296
CCC	0	0	0	0	0.06733	0.14355	0.74361	0.04552
D	0	0	0	0	0	0	0	1.00000
	TIME 15-18							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.94688	0.05050	0.00262	0	0	0	0	0
AA	0.00377	0.92617	0.07006	0	0	0	0	0
A	0	0.02797	0.93524	0.03576	0.00104	0	0	0
BBB	0.00215	0	0.04336	0.93813	0.01437	0.00199	0	0
BB	0	0	0.00497	0.08134	0.83046	0.06627	0.00643	0.01053
B	0	0	0.00188	0.00649	0.08795	0.84445	0.01090	0.04833
CCC	0	0	0	0	0.07399	0.07638	0.83271	0.01691
D	0	0	0	0	0	0	0	1.00000
	TIME 15-19							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.71135	0.20969	0.07895	0	0	0	0	0
AA	0.00329	0.95760	0.03911	0	0	0	0	0
A	0	0.02745	0.96394	0.00822	0.00038	0	0	0
BBB	0.00687	0	0.27043	0.58332	0.13605	0.00334	0	0
BB	0	0	0.01711	0.08730	0.83929	0.04680	0.00753	0.00197
B	0	0	0.00330	0.00732	0.05632	0.90751	0.02157	0.00397
CCC	0	0	0	0	0.16897	0.21855	0.51403	0.09845
D	0	0	0	0	0	0	0	1.00000

Table 4.8.2: probabilities $\varphi_{ij}(15,t)$

Tables 4.9.1 and **4.9.2** report the probabilities $\phi_{ij}(s,t)$.

$\phi_{ij}(s,t)$ EVOLUTION EQUATION MATRICES								
	TIME 0-1							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.99243	0.00757	0	0	0	0	0	0
AA	0.00004	0.99938	0.00059	0	0	0	0	0
A	0	0.00438	0.99303	0.00252	0.00007	0	0	0
BBB	0	0	0.00182	0.99560	0.00258	0	0	0

BB	0	0	0.00054	0.00178	0.96968	0.02790	0.00010	0
B	0	0	0.00019	0	0.00304	0.99543	0.00135	0
CCC	0	0	0	0	0	0.00549	0.99451	0
D	0	0	0	0	0	0	0	1.00000
	TIME 0-10							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.94504	0.05212	0.00233	0.00030	0.00016	0.00003	0.00001	0.00000
AA	0.01088	0.94684	0.03928	0.00209	0.00039	0.00045	0.00006	0.00002
A	0.00024	0.02331	0.93634	0.03691	0.00235	0.00073	0.00004	0.00008
BBB	0.00015	0.00072	0.03173	0.92934	0.03395	0.00299	0.00038	0.00074
BB	0.00001	0.00032	0.00569	0.02750	0.79898	0.15778	0.00483	0.00489
B	0.00001	0.00023	0.00649	0.00189	0.02921	0.93979	0.01409	0.00828
CCC	0.00013	0.00002	0.00022	0.00042	0.00178	0.05494	0.91981	0.02268
D	0	0	0	0	0	0	0	1.00000
	TIME 0-19							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.82509	0.14645	0.02217	0.00522	0.00062	0.00026	0.00004	0.00016
AA	0.02650	0.77411	0.17356	0.02017	0.00229	0.00206	0.00034	0.00097
A	0.00154	0.06670	0.78728	0.12462	0.01322	0.00449	0.00053	0.00162
BBB	0.00073	0.00777	0.11696	0.74792	0.09341	0.02094	0.00338	0.00890
BB	0.00052	0.00265	0.02266	0.11474	0.49705	0.27871	0.02436	0.05931
B	0.00023	0.00178	0.01625	0.02366	0.11988	0.68643	0.04612	0.10564
CCC	0.00122	0.00052	0.00712	0.01329	0.03557	0.19796	0.41307	0.33124
D	0	0	0	0	0	0	0	1.00000

Table 4.9.1: probabilities $\phi_{ij}(0,t)$

For example 0.03691 represents the probability to be in the state **BBB** at time 10, given that the rating evaluation was **A** at time 0.

$\phi_{ij}(s,t)$ EVOLUTION EQUATION MATRICES								
	TIME 15-16							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.99706	0.00257	0.00037	0	0	0	0	0
AA	0.00104	0.98569	0.01327	0	0	0	0	0
A	0	0.00431	0.98938	0.00600	0.00031	0	0	0
BBB	0.00037	0	0.00174	0.99296	0.00470	0.00024	0	0
BB	0	0	0.00173	0.00919	0.98279	0.00223	0.00210	0.00195
B	0	0	0.00073	0.00133	0.01972	0.96755	0.00357	0.00710
CCC	0	0	0	0	0.05100	0.05393	0.88217	0.01290
D	0	0	0	0	0	0	0	1.00000
	TIME 15-17							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.97884	0.01856	0.00258	0.00002	0.00000	0.00000	0	0
AA	0.00258	0.97692	0.01865	0.00154	0.00002	0.00030	0	0
A	0.00002	0.01303	0.97634	0.00993	0.00054	0.00013	0.00000	0.00000
BBB	0.00049	0.00045	0.02373	0.96076	0.01296	0.00137	0.00015	0.00010
BB	0	0.00002	0.00441	0.01362	0.94654	0.02774	0.00444	0.00322
B	0	0.00001	0.00209	0.00340	0.04955	0.91258	0.01195	0.02043

CCC	0	0	0.00021	0.00143	0.05455	0.07365	0.83737	0.03279
D	0	0	0	0	0	0	0	1.00000
	TIME 15-18							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.96944	0.02615	0.00396	0.00044	0.00000	0.00001	0.00000	0.00000
AA	0.00511	0.93798	0.05106	0.00474	0.00009	0.00100	0.00001	0.00001
A	0.00016	0.02467	0.95236	0.02107	0.00138	0.00032	0.00002	0.00002
BBB	0.00073	0.00075	0.04707	0.91985	0.02659	0.00360	0.00062	0.00079
BB	0.00067	0.00011	0.00851	0.06487	0.86584	0.04090	0.01243	0.00667
B	0.00008	0.00009	0.00382	0.00713	0.08085	0.85875	0.01694	0.03235
CCC	0.00006	0.00002	0.00429	0.00370	0.05295	0.15521	0.66296	0.12081
D	0	0	0	0	0	0	0	1.00000
	TIME 15-19							
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.92405	0.06701	0.00630	0.00258	0.00002	0.00001	0.00000	0.00003
AA	0.00693	0.87733	0.10567	0.00801	0.00032	0.00104	0.00004	0.00065
A	0.00028	0.04107	0.89584	0.05889	0.00259	0.00082	0.00004	0.00048
BBB	0.00143	0.00348	0.07969	0.85841	0.04557	0.00778	0.00092	0.00273
BB	0.00103	0.00104	0.01174	0.09936	0.75585	0.09407	0.01424	0.02267
B	0.00012	0.00045	0.00568	0.01449	0.11033	0.75180	0.03423	0.08291
CCC	0.00009	0.00020	0.00789	0.01098	0.07388	0.19592	0.52433	0.18672
D	0	0	0	0	0	0	0	1.00000

Table 4.9.2: probabilities $\phi_{ij}(15,t)$

The last **Table 4.10** reports the reliability probabilities giving the probabilities that a firm being in a given rating at time *s* will not have a default up to the time *t*.

RELIABILITY								
TIMES		**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
0	1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0	2	1.00000	1.00000	1.00000	0.99998	0.99996	0.99988	0.99941
0	3	1.00000	1.00000	1.00000	0.99995	0.99978	0.99973	0.99882
0	8	1.00000	0.99999	0.99996	0.99956	0.99773	0.99588	0.99008
0	9	1.00000	0.99999	0.99994	0.99945	0.99706	0.99456	0.98414
0	10	1.00000	0.99998	0.99992	0.99926	0.99511	0.99172	0.97732
0	17	0.99998	0.99986	0.99949	0.99621	0.97209	0.95248	0.83528
0	18	0.99997	0.99977	0.99924	0.99480	0.96292	0.93323	0.76184
0	19	0.99984	0.99903	0.99838	0.99110	0.94069	0.89436	0.66876
1	2	1.00000	1.00000	0.99991	0.99971	1.00000	0.99864	0.98889
1	11	1.00000	0.99994	0.99723	0.99736	0.96805	0.97156	0.82190
1	19	0.99984	0.99864	0.99397	0.98721	0.91388	0.85046	0.53408
2	3	1.00000	1.00000	1.00000	0.99978	0.99877	0.99834	1.00000
2	11	0.99999	0.99997	0.99990	0.99740	0.98844	0.96680	0.96643
2	19	0.99976	0.99887	0.99827	0.98681	0.94980	0.84825	0.69136
5	6	1.00000	1.00000	0.99992	0.99998	0.99940	0.99386	0.99340
5	13	1.00000	0.99987	0.99850	0.99663	0.98779	0.93928	0.83637

5	19	0.99990	0.99847	0.99482	0.98470	0.94955	0.78590	0.54287
7	8	1.00000	1.00000	1.00000	1.00000	0.99895	0.99744	0.99036
7	14	0.99997	0.99993	0.99982	0.99790	0.98951	0.95937	0.81924
7	19	0.99977	0.99884	0.99805	0.98807	0.95686	0.86042	0.55024
10	11	1.00000	1.00000	1.00000	0.99987	0.99629	0.98907	0.95349
10	15	1.00000	1.00000	0.99996	0.99658	0.98056	0.92538	0.76563
10	19	0.99989	0.99927	0.99906	0.98865	0.94983	0.79378	0.51705
13	14	1.00000	1.00000	0.99999	1.00000	0.99990	0.99821	0.98290
13	17	1.00000	0.99997	0.99926	0.99921	0.99593	0.97229	0.85015
13	19	0.99994	0.99886	0.99804	0.99625	0.97754	0.91261	0.60743
17	18	1.00000	1.00000	1.00000	0.99803	0.99899	0.96246	0.74975
17	19	0.99997	0.99926	0.99974	0.99495	0.97565	0.93488	0.55347

Table 4.10: reliability probabilities

Chapter 9

GENERALISED NON-HOMOGENEOUS MODELS FOR PENSION FUNDS AND MANPOWER MANAGEMENT

In this chapter, we present more applications of NHSMP to insurance, particularly in the field of pension funds, showing that non-homogeneous models can be useful in real-life applications, with realistic results based on scenarios that can be treated numerically, even if this involves new software.

1 APPLICATION TO PENSION FUNDS EVOLUTION

This model is a general, rigorous and tractable stochastic evolution time model for pension funds, called the discrete time non-homogeneous semi-Markov pension fund model, taking into account economic, financial and demographic evolution factors so that it becomes a real-life model.
The most important factors are: *seniority*, *general age dependence*, *rate of inflation and salary lines.*
To take into account all these aspects, the DTNHSMP defined in Chapter 4 will be generalized. We call this model Generalized Discrete Time Non-Homogeneous Semi-Markov Process (GDTNHSMP).
The model starts from a set of m states and each member of the fund is necessarily in one and only one of these states at each time epoch, for example each year.
The main probabilistic assumption is that the successive state transitions together with transition time epochs constitute a two-dimensional non-homogeneous Markov additive process on which the state at any time epoch t is defined by the imbedded NHSMP.
Let us say that we introduce as another fundamental tool the concept of a *scenario* for strategic choices of the considered company, or even a government, and that of an *economic scenario* for the impact of the future economic environment.
Finally, we also consider the *statistical estimation problem* of the two-dimensional semi-Markov kernel using internal and external data.

1.1 Introduction

Dynamic management of a pension fund is generally dealt with models, generally quite simple from the mathematical point of view (see Khorasanee (1994)) for the U.K. experience.
However, more elaborate theoretical stochastic models are now possible (see Dufresne (1986), (1988), Haberman (1994), Balcer and Sahin (1983)), mostly by time continuous models and with restrictive conditions to get real-life models.
Real-life models for the dynamics of pension funds are complicated (see Manly (1902), Myers (1988), Tomassetti (1973)) mainly because these models must work with a far off horizon and with many possible parameterizations. Moreover, there is a lack of sophisticated but tractable classes of stochastic processes to be used for building such models.
The model presented here was given in Janssen and Manca (1997a) and generalizes the DTNHSMP that was shown in Chapter 4. The DTNHSMP is based on the theoretical results of Janssen and De Dominicis (1984). Another paper on this topic that presented an algorithmic approach to the GDTMHSMP in the pension fund environment was by Janssen and Manca (1998).
The selection of discrete time models is quite natural, as we know that the management of a pension fund is mainly on a yearly basis. Moreover the model must be non-homogeneous in age, as all the parameters concerning the members of the fund are age dependent. Finally it is a semi-Markov model because we must consider transition states in connection with transition times.
The main parameterizations are:

(i) the introduction of time,
(ii) the introduction of inflation,
(iii) the time hedging dependence of payments from the fund and the premium dues paid to the fund,
(iv) the concept of a scenario giving in particular the possibility to model the flows of new entrances to the fund.

Let us mention that the non-homogeneous semi-Markov models, or more often only Markov models in pension schemes or in related environments, have been presented in many papers. We mention for example in manpower planning, Tsantas (1993) and Tsantas and Vassiliou (1993). Sahin (1993) also uses a particular non-homogeneous model as an extension of preceding homogeneous models from Sahin and Balcer (1979) and (1983). Let us recall that all these models are time continuous, giving thus an easier mathematical treatment than in models some reasonably predictable of discrete time.
In practice, the most important problem is to study the *dynamic financial equilibrium* of the fund. To do so, we have to compute the asset and liability flows for the whole future to get:

(i) the flow of reserves,
(ii) the equilibrium premiums.

Unfortunately, these premiums are always too high. This is due to the fact that the rules of modern day pension schemes are too "generous" and that there now exist in most developed countries a demographic evolution and an economic environment containing few active workers and more pensioners for a longer time. To find and maintain an acceptable equilibrium, there must exist, in our view, a new type of *solidarity* between successive generations involving not only public authorities but the active cooperation of the insurance companies themselves.
Then the fundamental question is: what the cost of this solidarity? We believe that the most promising strategy is to use simulation models involving changes of economic, financial and demographic parameters. This can be done by selection of a model that we have defined as a *scenario*. Although such models of course already exist (Tomassetti (1973), (1991), Volpe di Prignano and Manca (1988), Bacinello (1988),...), the GDTNHSM model presented here seems to give, as far as we know, the most general structure and flexibility in choosing basic parameter values for:

-rules of the pension fund,
-flows of premiums and pension amounts,
-seniority influence,
-rate of inflation forecasts,
-changes in the salary lines.

With this choice of scenario, the GDTNHSM pension fund model can provide a general framework for pricing solidarity proposals and for splitting their costs between public and private social insurance sources.

1.2 The Non-Homogeneous Semi-Markov Pension Fund Model

It is well known that the pension fund problem is one of the most important problems of the present time, not only for people today but for future generations.
Clearly, this problem must be placed in a general economic, financial, demographic and political framework. For example, one of the basic facts is a change in mortality rates: in almost all countries, these rates are decreasing so that more and more people will be entitled to a pension.

It is now a fact that the numbers of the working population will also decrease. Nowadays, most national governments are preoccupied with the catastrophic evolution of national pension funds and some now see a need for collaboration with insurance companies. In any event, whatever the future choice of such collaboration may be, we will always need actuarial models that will describe the stochastic evolution of pension funds.
To be realistic enough, these models must depend on many of parameters and particularly may be non-homogeneous in time for obvious reasons like the ones mentioned above. Moreover, as it is generally impossible to predict the evolution of basic parameters on salary evolution, on inflation, on disability and so on, these models must be able to study the influence of possible *scenarios* in order to hedge against undesirable changes.
For example, within a selected scenario, we can use asset liability management techniques to preserve the financial equilibrium of the fund. We can also study the possible impact of a new demographic development or that of changes in mortality rates or also the impact of a manpower expansion of the society concerned, etc.
The model presented here offers all these possibilities. To give a clear understanding of our model, we will proceed in two parts: first, we will show how we manage time non-homogeneity with DTNHM in this environment and second, we will present way to introduce the possible influences of time evolution of demography and salaries, taking into account the basic rules of the considered fund.
For simplicity, we present the model for one selected company or society but note it is also possible to consider the same type of model on a macroeconomic level provided we have enough data.
The pension fund model should generalize the DTNHSMP presented in Chapter 4. In this way it is possible to take into account all the different aspects that are important to follow the time evolution of a pension scheme. The generalization will be made step by step, introducing each time a new temporal variable.
For a better understanding of the generalization of the different steps, we will also repeat the introduction of the DTNHSMP that represents the initial one.

1.2.1. The DTNHSM Model

One of the simplest models for pension time evolution uses a four state space model with a as active state, i for the invalidity state, p for pension state and d for death or outgoing state. Let us denote this state space by I with

$$I = \{a, i, p, d\} . \tag{1.1}$$

Clearly in the simplest case, at any time n, each member of the pension fund is in one and only one of these four states.
In one time unit, some transitions are possible and some others are not and of course, state d is clearly an absorbing state.

More generally, we will now suppose that the state space has m elements:

$$I=\{1,...,m\}. \tag{1.2}$$

Let us now introduce a *discrete time scale*: we observe the state at times 0,1,2,...,n,....

If the random variable J_n represents the state of the member at transition n, it is usual to assume that the discrete time stochastic process $(J_n, n \geq 0)$ is a homogeneous Markov chain with $\mathbf{P} = \left[p_{ij} \right]$ as transition matrix.

However, it is much more realistic to introduce age dependence for this matrix: if s represents the *member age* at the transition n, the transition probability matrix is now written as

$$\mathbf{P}(s) = \left[p_{ij}(s) \right] \tag{1.3}$$

and consequently, the stochastic process $(J_n, n \geq 0)$ becomes non-homogeneous in age.

To get a more realistic model, we will now introduce another random sequence $(T_n, n \geq 0)$, T_n representing the age of the member at transition n.

We do not only want to study *closed pension funds* for which all members are present at time 0 and nobody else can enter later, but also *open pension funds* including the possibility of adding new members at any time t .This means that if we select a new member at time n, we must first observe not only his state but also his age.

So, we can now introduce the two-dimensional stochastic process

$$((J_n, T_n), n \geq 0) \tag{1.4}$$

where clearly:

$$0 \leq T_0 \leq T_1 \leq \cdots \leq T_n \leq \cdots, \tag{1.5}$$

T_0 representing the entrance time of the new member.

The basic assumption is that the (J,T) process is a discrete time non-homogeneous Markov renewal process so that (see Chapter 4):

$$\begin{aligned} &P\left(J_{n+1} = j, T_{n+1} \leq t \middle| J_\nu, T_\nu, \nu \leq n-1, J_n = i, T_n = s\right), \\ &P\left(J_{n+1} = j, T_{n+1} \leq t \middle| J_n = i, T_n = s\right), \end{aligned} \tag{1.6}$$

with of course $s < t$, an assumption which seems quite natural for pension fund time evolution.

As usual, the associated non-homogeneous semi-Markov matrix kernel $\mathbf{Q}$ is defined as the $m \times m$ matrix whose general element is given by:

$$Q_{ij}(s,t) = P\left(J_{n+1} = j, T_{n+1} \leq t \middle| J_n = i, T_n = s\right). \tag{1.7}$$

As the time scale is discrete, the general element of the matrix $\mathbf{B}$ is:

$$b_{ij}(s,t) = P\left(J_{n+1} = j, T_{n+1} = t \middle| J_n = i, T_n = s\right), \tag{1.8}$$

and so

$$Q_{ij}(s,t)=\sum_{h=s}^{t} b_{ij}(s,t)\,. \tag{1.9}$$

It is now possible to express the matrix $\mathbf{P}(s)=\left[\,p_{ij}(s)\right]$ as follows:

$$p_{ij}(s)=P\left(J_{n+1}=j\,\middle|\,J_n=i,T_n=s\right) \tag{1.10}$$

and so with relation (1.7):

$$p_{ij}(s)=Q_{ij}(s,\infty)\,. \tag{1.11}$$

For the sequel, we also need to introduce the following conditional marginal probability:

$$H_i(s,t)=P\left(T_{n+1}\le t\,\middle|\,J_n=i,T_n=s\right) \tag{1.12}$$

giving the probability that the member will leave state i before or at age t.
Of course, we also have:

$$H_i(s,t)=\sum_{j=1}^{m} Q_{ij}(s,t), \tag{1.13}$$

$$H_i(s,\infty)=1. \tag{1.14}$$

Another interesting random variable is the *sojourn time* in one of the m states just after a transition in this state: here too, if at time s, the new member is entering in state i, the probability that he will leave this state before t with a transition to state j will be represented by the function $F_{ij}(s,t)$ and we know that

$$F_{ij}(s,t)=\frac{Q_{ij}(s,t)}{p_{ij}(s)}, \tag{1.15}$$

provided that, of course, the probability $p_{ij}(s)$ is strictly positive.

The last probabilities will be of great interest for the statistical estimation problem described later.
Of course, one of the main interests of the proposed model is related to the definition of the so-called associated non-homogeneous semi-Markov process $Z=(Z(t),t\ge 0)$ representing, for each time t, the state occupied by the member for which the transition probabilities will be written as

$$\phi_{ij}(s,t)=P\left(Z(t)=j\,\middle|\,Z(s)=i\right), \tag{1.16}$$

which are solutions of the algebraic system:

$$\phi_{ij}(s,t)=(1-H_i(s,t))\delta_{ij}+\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\phi_{kj}(\vartheta,t)\,. \tag{1.17}$$

In this case the model presented here will be called the *discrete time non-homogeneous semi-Markov pension funds model* (in short DTNHSMPFM).

1.2.2. The States Of DTNHSMPFM

For any stochastic model, it is very important to select a set of states not only in a parsimonious way but also to give the best possible description of the dynamic time evolution of the considered system.
Every pension fund depends on a written contract called the *pension scheme*; the nature of different states is in fact reflected by this scheme.
Since we would like our model to be applicable to a wide variety of pension funds, , we propose the following selection for the *m* states of *I*:

- the first $m-5$ states: $1,2,\ldots,m-5$ give all possible worker states, i.e., the different job ranks within the considered firm,
- state $m-4$ represents the disability state,
- state $m-3$ represents the pension state, taken at the usual age written in the pension scheme,
- state $m-2$ represents the pre-pension state, taken at the permitted age written in the pension scheme but before the age of the normal pension,
- state $m-1$ represents the survivor pension state, given until the permitted age written in the pension scheme to survivors after the death of the member,
- state *m* represents the absorbing state of leaving the fund with no more charges due to membership, for example death without any survivor:

The graph of possible transitions is given in **Figure 1.1**.
Of course, we may simplify or complicate our model with the suppression or addition of other states to get a particular DTNHSMPFM.
For example, we begin this section with the presentation of a four-states model which is the simplest possible model. We could of course subdivide state $m-4$ in two states, *illness* and *disability*, and furthermore consider the different disability degrees.
From the theoretical and numerical points of view, such an introduction of some supplementary states does not raise a problem. However, from the practical point of view, this introduction complicates the delicate *statistical problem* of estimating the necessary data to have an operational model.

1.2.3 The Concept Of Seniority In The DTNHSMPFM

When a member of the fund is in one of the first $m-5$ states and has a potential for a transition to one of the last five states, we must add information to be able to evaluate the *financial charges* for the fund and moreover, for a sojourn in the $(m-5)$ first states, we need to evaluate the *incomes* to the fund.
As both financial charges and incomes of the fund depend on the salaries of the members, the necessary basic information is related to the salary scale through the concept of *seniority*.

The *seniority* concept represents the time spent by a member in the company since his first entrance. As often it is possible to have fictive seniority at the entrance time of the member, for example to attract competent members in specialized fields, this initial seniority will be represented by the non-negative random variable S_0.

We give some constraints to seniority, more precisely:

- seniority cannot be greater then 0 at an initial work age α, for example 18,
- in a first statement of our model, seniority continues to increase also after retirement.

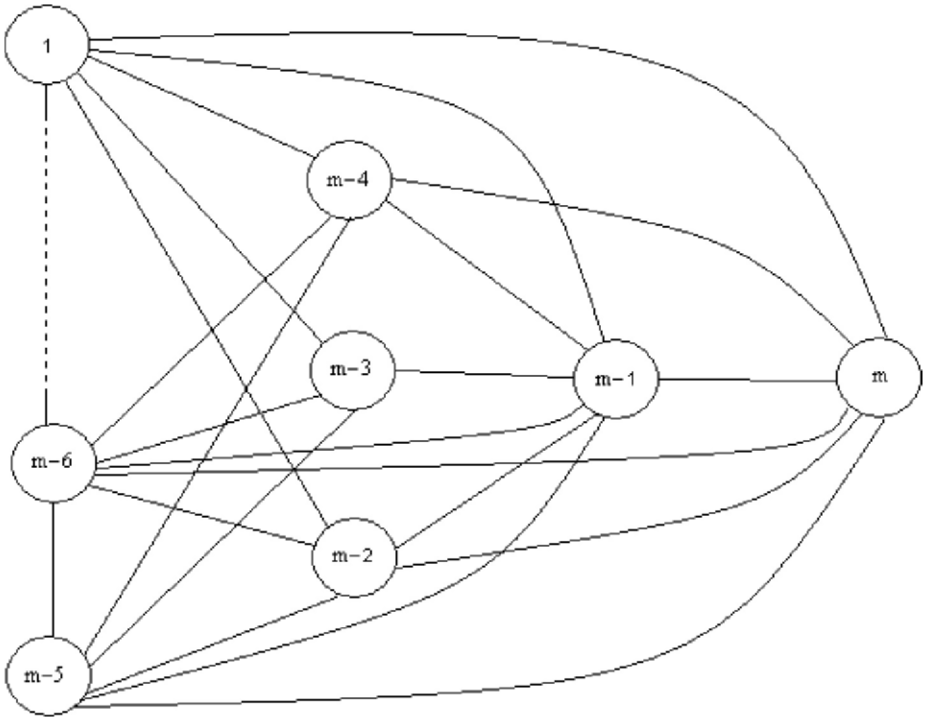

Figure 1.1: pension fund transitions

The second constraint can appear wrong, but once the pension is fixed (at retirement age) there are no financial influences on the fund, moreover in this way we know the fictive number of years that a person has in the fund.

In the following we will also present a version of the model with maximum seniority. This model will be more tortuous then the one that we are going to present.

The seniority $S_n, n \geq 1$ is usually defined by the relation:

$$S_{n+1} = S_n + T_{n+1} - T_n, n > 0. \quad (1.18)$$

For some members, it may however be necessary to add another fictive seniority, due to an *exceptional promotion* for example.
The introduction of the new stochastic process $S=(S_n, n\geq 1)$ implies that now, at each state transition time *n*, the considered member of the fund is characterized by the triple $(J_n, T_n, S_n), n\geq 1$.
Our new assumption will be that this (*J,T,S*) process is a *bi-dimensional non-homogeneous Markov renewal process* (*J*,(*T*-*S*)) with as kernel:

$$^{\tau}Q_{ij}(s,t)=P(J_{n+1}=j, T_{n+1}\leq t, S_{n+1}\leq \tau+t-s \mid J_n=i, T_n=s, S_n=\tau). \quad (1.19)$$

Probabilities defined in the preceding section only for the (*J,T*) process may be easily extended to the (*J,T,S*) process as follows:

$$^{\tau}b_{ij}(s,t)=P(J_{n+1}=j, T_{n+1}=t, S_{n+1}=\tau+t-s \mid J_n=i, T_n=s, S_n=\tau) \quad (1.20)$$

so that:

$$^{\tau}Q_{ij}(s,t)=\sum_{h=s}^{t}{}^{\tau}b_{ij}(s,h). \quad (1.21)$$

Similarly, if we define:

$$p_{ij}(s)=P\left(J_{n+1}=j \mid J_n=i, T_n=s\right) \quad (1.22)$$

we have:

$$^{\tau}p_{ij}(s)={}^{\tau}Q_{ij}(s,\infty). \quad (1.23)$$

These other following extensions are straightforward:

$$^{\tau}H_i(s,t)=\sum_{j=1}^{m}{}^{\tau}Q_{ij}(s,t) \quad (1.24)$$

where:

$$^{\tau}H_{i(st)}=P\left(T_{n+1}\leq t \mid J_n=i, T_n=s, S_n=\tau\right), \quad (1.25)$$

$$^{\tau}H_i(s,\infty)=1, \quad (1.26)$$

$$^{\tau}F_{ij}(s,t)=\frac{^{\tau}Q_{ij}(s,t)}{^{\tau}p_{ij}(s)}, \quad (1.27)$$

where

$$^{\tau}F_{ij}(s,t)=P\left(T_{n+1}\leq t \mid J_n=i, J_{n+1}=j, T_n=s, S_n=\tau\right), \quad (1.28)$$

i.e. the *sojourn time conditional distribution* entering in state *i* at time transition *n* with a seniority τ.
Finally, it is also possible to define the *two-dimensional semi-Markov process* associated with the two-dimensional NHMRP (*J*,(*T*-*S*)) noted as

$$\overline{Z}=({}^{\tau}Z(t),\ t,\tau\geq 0), \quad (1.29)$$

$^{\tau}Z(t)$ representing, for each time t, the state occupied by the member of seniority τ.
The transition probabilities of this last process are given by:

$$ {}^{\tau}\phi_{ij}(s,t) = P\left({}^{\tau+t-s}Z(t) = j \,\middle|\, {}^{\tau}Z(s) = i \right) \tag{1.30} $$

satisfying the analog system complying with (1.17):

$$ {}^{\tau}\phi_{ij}(s,t) = (1 - {}^{\tau}H_i(s,t))\delta_{ij} + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} {}^{\tau}b_{ik}(s,\vartheta)\, {}^{\tau+\vartheta-s}\phi_{kj}(\vartheta,t). \tag{1.31} $$

To conclude this section, we can say that:
- the introduction of the concept of seniority is mathematically tractable,
- the introduction of seniority represents the first generalization step,
- the problem of getting data will be studied later.

1.3 The Reserve Structure

To apply this model to the pension fund problem we need to consider a *reward structure* connected with the semi-Markov process representing the financial charges and incomes of the considered fund.

Let us define now:

$V_i(s,t)$: the discounted expected reserve at a fixed epoch of the reward in the time interval $[s,t)$, given that there is an entrance in state *i* at age *s*,

ψ_i: the amount paid per time period in state *i* (permanence reward),

r: the fixed rate of interest,

$a_{\overline{h}|r}$: the present value of a unitary h-period annuity i.e.:

$$ a_{\overline{h}|r} = \sum_{k=1}^{h} (1+r)^{-k}. \tag{1.32} $$

The evolution equations are the following ones:

$$ \begin{aligned} V_i(s,t) &= (1 - H_i(s,t))\psi_i a_{\overline{t-s}|r} + \sum_{\theta=s+1}^{t} \sum_{\beta=1}^{m} b_{i\beta}(s,\theta)\psi_i a_{\overline{\theta-s}|r} \\ &+ \sum_{\theta=s+1}^{t} \sum_{\beta=1}^{m} b_{i\beta}(s,\theta) V_\beta(\theta,t)(1+r)^{s-\theta}, \end{aligned} \tag{1.33} $$

$i = 1,...,m; \quad 0 \le s \le t; \quad s,t = 0,1,....$

In this case, the $V_i(s,t)$ are the discounted expected values of the reserves that have been paid from *s* to *t* when a member has arrived in state *i* at age *s*.

The term $(1 - H_i(s,t))$ represents the probability to remain in state *i* once a member has arrived there at age *s*, a member having to pay (or to get) at each period of time the reward ψ_i. This first term represents, as already explained in Chapter 4, the expected value of this amount.

The second term of (1.33) gives the expected present value of the rewards that a member arrived in *i* at age *s* paid in this state before changing the state.

The last term of relation (1.33) represents the expected value of the reserves that a member who arrived in state *i* at age *s* and changed his situation at age θ has to

pay in the other state. These values are discounted at time θ, so we need to discount them at time *s*.
As with the evolution equations (1.33) it is not possible to allow for different behaviors as a function of the seniority of people. We need to change the evolution equations of the reserve process to introduce a seniority factor, generalizing also the reward process, as we did for the semi-Markov process.
In this light, the relations (1.33) become:

$$
\begin{aligned}
{}^{\tau}V_i(s,t) &= (1-{}^{\tau}H_i(s,t))\,{}^{\tau}\psi_i a_{\overline{t-s}|r} + \sum_{\theta=s+1}^{t}\sum_{\beta=1}^{m} {}^{\tau}b_{i\beta}(s,\theta)\,{}^{\tau}\psi_i a_{\overline{\theta-s}|r} \\
&+ \sum_{\theta=s+1}^{t}\sum_{\beta=1}^{m} {}^{\tau}b_{i\beta}(s,\theta)\,{}^{\tau+\theta-s}V_\beta(\theta,t)(1+r)^{s-\theta}.
\end{aligned}
\tag{1.34}
$$

The meaning of (1.34) is analogous to that of (1.33), the only difference being that now it is moreover possible to consider seniority. In this way, the probabilities of changing states because of seniority, and moreover, it is possible to consider also different rewards as a function of different seniorities namely:

$$
{}^{\tau}\psi_i, \tau \geq 0, i = 1,\ldots,m. \tag{1.35}
$$

1.4. The Impact Of Inflation And Interest Variability

To begin with, let us point out that it is important to make some assumptions on the *moments* of the reward payment and of the state changes, because we are working with discrete time models.
Our main assumptions are the following:

i) amounts of money are paid entirely at the midpoint of our period (we can suppose that the period is one year, for example),
ii) changes always happen at the midpoint but after the reward payment.

Figure 1.2 illustrates this assumption.

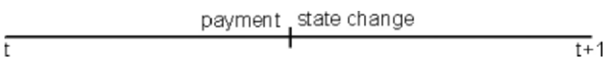

Figure 1.2: payment and state change instants

In general, rewards are also time dependent but we still assume that it is not the case, because the evolution equations of DTNHSMP are not involved in this change. We only suppose that the transition probabilities change in time but only because of age and seniority.
Another important point is to take into account both *inflation* and *interest rate variability,* the latter being represented by the variations of the yield curve so that

the model can measure the impact of the *interest risk* or of a change of an interest scenario.
Let us thus introduce:

$$r_i > 0, i = 1, ..., w, \tag{1.36}$$

representing the successive interest rates in our time horizon, $[0, w]$ being the time interval that we consider for the simulation.
From these values, taking into account the hypotheses that we made, we can construct the present value factors.
Let us be precise that we need to consider two different kinds of factors: one, ν', for the present value of the reward and the other, ν for discounting the ${}^{\tau}V_i(s,t)$, $r_h > 0, h = 1, ..., w.$
Furthermore we now can introduce for the amount ${}^{\tau}\psi_i$ a new dependence on the *time h*. We will note:

$${}^{\tau}\psi_i(h), \quad \tau \geq h, i = 1, \ldots, m; h = 0, \ldots, w. \tag{1.37}$$

Figure 1.3 describes the reward payment process.
Figures 1.4 and **1.5** describe the two different discounting processes and after them the related formulas are reported:

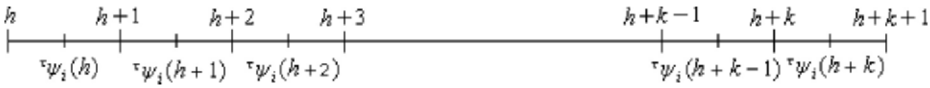

Figure 1.3: reward payments

Figure 1.4: discount factors of reward payments

Figure 1.5: discount factors of reward process

We need a different discounting process. Indeed, the ${}^{\tau}V_i(s,t)$ are discounted always at the initial moment and the related discount factors will be $\nu(s,t)$. The sums received or paid by the fund instead will be discounted by the $\nu'(s,t)$ considering one half period more.
After these assumptions that also take into account the passage of time, the evolution equations for a discrete time non-homogeneous semi-Markov pension reserve process DTNHSMPRP are:

$$
{}^{\tau,h}V_i(s,t) = (1 - {}^{\tau}H_i(s,t)) \sum_{k=h}^{h+t-s-1} {}^{\tau+k-h}\psi_i(k)\nu'(h,k)
$$
$$
+ \sum_{\theta=s+1}^{t} \sum_{\beta=1}^{m} {}^{\tau}b_{i\beta}(s,\theta) \sum_{k=h}^{h+\theta-s-1} {}^{\tau+k-h}\psi_i(k)\nu'(h,k) \tag{1.38}
$$
$$
+ \sum_{\theta=s+1}^{t} \sum_{\beta=1}^{m} {}^{\tau}b_{i\beta}(s,\theta)\, {}^{\tau+\theta-s,h+\theta-s}V_\beta(\theta,t)\nu(h,h+\theta-s)
$$

with

$$
\nu(h,k) = \prod_{\alpha=h+1}^{h+k} (1+r_\alpha)^{-1}, \;\; \nu(h,h) = 1, \tag{1.39}
$$

$$
\nu'(h,k) = \prod_{\alpha=h}^{h+k-1} (1+r_\alpha)^{-1}(1+r_{h+k})^{-.5}. \tag{1.40}
$$

Let us say once more that the reserve ${}^{\tau,h}V_i(s,t)$ represents the mean present value at time *h* of all the rewards that were paid to members of seniority τ of age *s* at time *h* up to age *t*.
The next figure gives support for an example to explain formula (1.38).

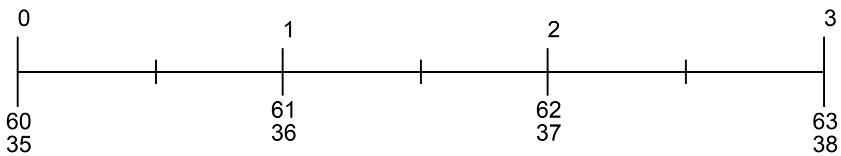

Figure 1.6: time axes

$$
{}^{35}H_i(60,62) = \sum_{\theta=60}^{62} \sum_{j=1}^{m} {}^{35}b_{ij}(60,\theta), \tag{1.41}
$$

$$
{}^{35,0}V_i(60,62) = (1 - {}^{35}H_i(60,62)) \sum_{k=0}^{1} {}^{35+k}\psi_i(k)\nu'(0,k)
$$
$$
+ \sum_{\beta=1}^{m} {}^{35}b_{i\beta(60,61)} \sum_{k=0}^{0} {}^{35+k}\psi_i(k)\nu'(0,k) + \sum_{\beta=1}^{m} {}^{35}b_{i\beta}(60,62) \sum_{k=0}^{1} {}^{35+k}\psi_i(k)\nu'(0,k) \tag{1.42}
$$
$$
+ \sum_{\beta=1}^{m} {}^{35}b_{i\beta}(60,61)\, {}^{36,1}V_\beta(61,62)\nu(0,1) + \sum_{\beta=1}^{m} {}^{35}b_{i\beta}(60,62)\, {}^{37,2}V_\beta(62,62)\nu(0,2).
$$

1.5. Solving Evolution Equations

The equations (1.32) and (1.38) can be written in matrix form as follows:

$$
{}^{\tau}\mathbf{\Phi}(s,t) = (\mathbf{I} - {}^{\tau}\mathbf{H}(s,t)) + \sum_{\theta=s+1}^{t} {}^{\tau}\mathbf{B}(s,\theta)\, {}^{\tau+\theta-s}\mathbf{\Phi}(\theta,t), \tag{1.43}
$$

$$\begin{aligned}{}^{\tau,h}\mathbf{V}(s,t) &= (\mathbf{I} - {}^{\tau}\mathbf{H}(s,t)) \sum_{k=h}^{h+t-s-1} {}^{\tau+k-h}\boldsymbol{\psi}(k)\nu'(h,k) \\ &+ \sum_{\theta=s+1}^{t} {}^{\tau}\mathbf{B}(s,\theta) \sum_{k=h}^{h+\theta-s-1} {}^{\tau+k-h}\boldsymbol{\psi}(k)\nu'(h,k) \\ &+ \sum_{\theta=s+1}^{t} {}^{\tau}\mathbf{B}(s,\theta)\, {}^{\tau+\theta-s,h+\theta-s}\mathbf{V}(\theta,t)\nu(h,h+\theta-s),\end{aligned} \quad (1.44)$$

where ${}^{\tau}\mathbf{\Phi}(s,t), {}^{\tau}\mathbf{H}(s,t), {}^{\tau}\mathbf{B}(s,t)$ are square $m \times m$ matrices, ${}^{\nu,h}\mathbf{V}(s,t)$ an m vector and where:

$$0 \le \tau \le s \le t\,, \quad \tau, s, t, h \in \mathbb{N}\,, \quad (1.45)$$

$${}^{\tau}\mathbf{H}(s,t) = \left[\delta_{ij}(1 - H_i(s,t))\right], \quad (1.46)$$

$${}^{\tau+k}\boldsymbol{\psi}(k) = \begin{bmatrix} {}^{\tau+k}\psi_1(k) \\ {}^{\tau+k}\psi_2(k) \\ \vdots \\ {}^{\tau+k}\psi_m(k) \end{bmatrix}. \quad (1.47)$$

If we want to solve the two evolution equations for a finite time horizon considering the same period for both random variables S_n and T_n, i.e. we are considering w periods, we have:

$$0 \le \tau \le s \le t \le \omega; \quad h \le w \quad (1.48)$$

instead of (1.46) and then formulas (1.44) and (1.45) hold; ω represents the maximum reachable age.
The particular structure of systems (1.46) and (1.45) implies that no matrix inversion is necessary to get the solutions. In fact, in the case of (1.44) we have:

$${}^{w}\mathbf{\Phi}(\omega,\omega) = \mathbf{I}\,, \quad (1.49)$$

$${}^{w-1}\mathbf{\Phi}(\omega,\omega) = \mathbf{I}\,, \quad (1.50)$$

$${}^{w-1}\mathbf{\Phi}(\omega-1,\omega-1) = \mathbf{I}\,, \quad (1.51)$$

$${}^{w-1}\mathbf{\Phi}(\omega-1,\omega) = {}^{w-1}\mathbf{B}(\omega-1,\omega)\,{}^{w}\mathbf{\Phi}(\omega,\omega) + (\mathbf{I} - {}^{w-1}\mathbf{H}(\omega-1,\omega))\,. \quad (1.52)$$

The last relation gives the value of ${}^{w-1}\mathbf{\Phi}(\omega-1,\omega)$ and as:

$${}^{w-2}\mathbf{\Phi}(\omega,\omega) = \mathbf{I}\,, \quad (1.53)$$

$${}^{w-2}\mathbf{\Phi}(\omega-1,\omega-1) = \mathbf{I}\,, \quad (1.54)$$

$${}^{w-2}\mathbf{\Phi}(\omega-1,\omega) = {}^{w-2}\mathbf{B}(\omega-1,\omega)\,{}^{w-1}\mathbf{\Phi}(\omega,\omega) + (\mathbf{I} - {}^{w-2}\mathbf{H}(\omega-1,\omega))\,, \quad (1.55)$$

$${}^{w-2}\mathbf{\Phi}(\omega-2,\omega-2) = \mathbf{I}\,, \quad (1.56)$$

we find the following "backward" values:

$$\begin{aligned}{}^{w-2}\mathbf{\Phi}(\omega-2,\omega-1) &= {}^{w-2}\mathbf{B}(\omega-2,\omega-1)\,{}^{w-1}\mathbf{\Phi}(\omega-1,\omega-1) \\ &+(\mathbf{I} - {}^{w-2}\mathbf{H}(\omega-2,\omega-1)),\end{aligned} \quad (1.57)$$

$$\begin{aligned} {}^{w-2}\mathbf{\Phi}(\omega-2,\omega) &= {}^{w-2}\mathbf{B}(\omega-2,\omega-1)\,{}^{w-1}\mathbf{\Phi}(\omega-1,\omega) \\ &+ {}^{w-2}\mathbf{B}(\omega-2,\omega)\,{}^{w}\mathbf{\Phi}(\omega,\omega) + (\mathbf{I} - {}^{w-2}\mathbf{H}(\omega-1,\omega)). \end{aligned} \tag{1.58}$$

To go on with this special case of "backward substitution" we finally obtain:

$${}^{0}\mathbf{\Phi}(\alpha,\alpha) = \mathbf{I}, \tag{1.59}$$

$${}^{0}\mathbf{\Phi}(\alpha,\alpha+1) = {}^{0}\mathbf{B}(\alpha,\alpha+1)\,{}^{1}\mathbf{\Phi}(\alpha+1,\alpha+1) + (\mathbf{I} - {}^{0}\mathbf{H}(\alpha,\alpha+1)), \tag{1.60}$$

$$\begin{aligned} {}^{0}\mathbf{\Phi}(\alpha,\alpha+2) &= {}^{0}\mathbf{B}(\alpha,\alpha+1)\,{}^{1}\mathbf{\Phi}(\alpha+1,\alpha+2) \\ &+ {}^{0}\mathbf{B}(\alpha,\alpha+2)\,{}^{2}\mathbf{\Phi}(\alpha+2,\alpha+2) + (\mathbf{I} - {}^{0}\mathbf{H}(\alpha,\alpha+2)), \end{aligned} \tag{1.61}$$

$${}^{0}\mathbf{\Phi}(\alpha,\omega) = \sum_{\theta=\alpha+1}^{\omega} {}^{0}\mathbf{B}(\alpha,\theta)\,{}^{\theta-\alpha}\mathbf{\Phi}(\theta,\omega) + (\mathbf{I} - {}^{0}\mathbf{H}(\alpha,\omega)). \tag{1.62}$$

Now we will show how this special kind of "backward substitution" proceeds. We have a five indices matrix, this matrix can be seen as a three-dimensional matrix whose elements are m-order square matrices.

In **Figure 1.7** is shown the case in which $\omega - \alpha = 5$ and the number near the matrix represents seniority. For each seniority, we have a block matrix. First of all we can find the value of the unique element that is in the block matrix 5. Once we know this element we can find the three elements of the matrix 4; working on these elements with a "backward substitution", in the same way it is possible to compute the elements of the block matrix 3 and so on up to the matrix indexed by 0.

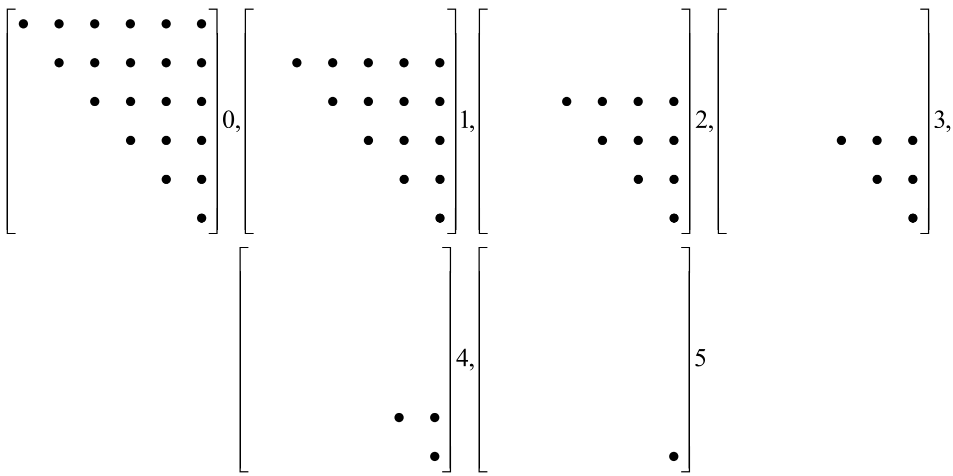

Figure 1.7: the GDTNHSMP backward substitution

Equations (1.45) have the same structure and so can be solved by a backward substitution, but with different indexes; for this reason we think it would be better to show also in this case how this process works.

We refer to the example shown at the end of the previous paragraph. In the following are described the matrix formulas of the "backward substitution" for this case.

$$^{w,0}\mathbf{V}(\omega,\omega)=\mathbf{0}, \tag{1.63}$$

$$^{w,1}\mathbf{V}(\omega,\omega)=\mathbf{0}, \tag{1.64}$$

$$^{w-1,0}\mathbf{V}(\omega-1,\omega-1)=\mathbf{0}, \tag{1.65}$$

$$\begin{aligned}^{w-1,0}\mathbf{V}(\omega-1,\omega)&=(\mathbf{I}-{}^{w-1}\mathbf{H}(\omega-1,\omega))\sum_{k=0}^{0}{}^{w-1}\boldsymbol{\psi}(0)\nu'(0,0)\\&+{}^{w-1}\mathbf{B}(\omega-1,\omega)\,{}^{w-1}\boldsymbol{\psi}(0)\nu'(0,0)+{}^{w-1}\mathbf{B}(\omega-1,\omega)\,{}^{w,1}\mathbf{V}(\omega,\omega)\nu(0,1).\end{aligned} \tag{1.66}$$

Using now result (1.64), we solve (1.66).
For the following steps we have:

$$^{w,2}\mathbf{V}(\omega,\omega)=\mathbf{0}, \tag{1.67}$$

$$^{w-1,1}\mathbf{V}(\omega-1,\omega-1)=\mathbf{0}, \tag{1.68}$$

$$\begin{aligned}^{w-1,1}\mathbf{V}(\omega-1,\omega)&=(\mathbf{I}-{}^{w-1}\mathbf{H}(\omega-1,\omega))\sum_{k=1}^{1}{}^{w-1}\boldsymbol{\psi}(k)\nu'(1,k)\\&+{}^{w-1}\mathbf{B}(\omega-1,\omega)\,{}^{w-1}\boldsymbol{\psi}(1)\nu'(1,1)+{}^{w-1}\mathbf{B}(\omega-1,\omega)\,{}^{w,2}\mathbf{V}(\omega,\omega)\nu(1,2),\end{aligned} \tag{1.69}$$

$$^{w-2,0}\mathbf{V}(\omega-2,\omega-2)=\mathbf{0}, \tag{1.70}$$

$$\begin{aligned}^{w-2,0}\mathbf{V}(\omega-2,\omega-1)&=(\mathbf{I}-{}^{w-2}\mathbf{H}(\omega-2,\omega-1))\sum_{k=0}^{0}{}^{w-1}\boldsymbol{\psi}(k)\nu'(0,k)\\&+{}^{w-2}\mathbf{B}(\omega-2,\omega-1)\,{}^{w-2}\boldsymbol{\psi}(0)\nu'(0,0)\\&+{}^{w-2}\mathbf{B}(\omega-2,\omega-1)\,{}^{w-1,1}\mathbf{V}(\omega-1,\omega-1)\nu(0,1),\end{aligned} \tag{1.71}$$

$$\begin{aligned}^{w-2,0}\mathbf{V}(\omega-2,\omega)&=(\mathbf{I}-{}^{w-2}\mathbf{H}(\omega-2,\omega))\sum_{k=0}^{1}{}^{w-2+k}\boldsymbol{\psi}(k)\nu'(0,k)\\&+\sum_{\theta=\omega-1}^{\omega}{}^{w-2}\mathbf{B}(\omega-2,\theta)\sum_{k=0}^{\theta-\omega+1}{}^{w-2+k}\boldsymbol{\psi}(k)\nu'(0,k)\\&+\sum_{\theta=\omega-1}^{\omega}{}^{w-2}\mathbf{B}(\omega-2,\theta)\,{}^{w+\theta+\omega,\theta-\omega+2}\mathbf{V}(\theta,\omega)\nu(0,\theta-\omega+2).\end{aligned} \tag{1.72}$$

In this way we can get the solutions of (1.72) knowing the previous results.
At last let us describe the solution for people that were at age α and seniority equal to 0 at the beginning of the simulation. We have successively:

$$^{w,w}\mathbf{V}(\omega,\omega)=\mathbf{0}, \tag{1.73}$$

$$^{w-1,w-1}\mathbf{V}(\omega-1,\omega-1)=\mathbf{0}, \tag{1.74}$$

$$
\begin{aligned}
&{}^{w-1,w-1}\mathbf{V}(\omega-1,\omega)=(\mathbf{I}-{}^{w-1}\mathbf{H}(\omega-1,\omega))\sum_{k=w-1}^{w-1}{}^{w-1}\boldsymbol{\psi}(k)\nu'(w-1,k)\\
&+{}^{w-1}\mathbf{B}(\omega-1,\omega)\,{}^{w-1}\boldsymbol{\psi}(w-1)\nu'(w-1,w-1)\\
&{}^{w-1}\mathbf{B}(\omega-1,\omega)\,{}^{w-1,w-1}\mathbf{V}(\omega-1,\omega)\nu(w-1,w),
\end{aligned}
\tag{1.75}
$$

and at last

$$
{}^{0,0}\mathbf{V}(\alpha,\alpha)=\mathbf{0}\,, \tag{1.76}
$$

$$
\begin{aligned}
&{}^{0,0}\mathbf{V}(\alpha,\alpha+1)=(\mathbf{I}-{}^{0}\mathbf{H}(\alpha,\alpha+1))\sum_{k=0}^{0}{}^{k}\boldsymbol{\psi}(k)\nu'(0,k)\\
&+{}^{0}\mathbf{B}(\alpha,\alpha+1)\,{}^{0}\boldsymbol{\psi}(0)\nu'(0,0)+{}^{0}\mathbf{B}(\alpha,\alpha+1)\,{}^{1,1}\mathbf{V}(\alpha,\alpha+1)\nu(0,1),
\end{aligned}
\tag{1.77}
$$

$$
\begin{aligned}
&{}^{0,0}\mathbf{V}(\alpha,\alpha+2)=(\mathbf{I}-{}^{0}\mathbf{H}(\alpha,\alpha+2))\sum_{k=0}^{1}{}^{k}\boldsymbol{\psi}(k)\nu'(0,k)\\
&+\sum_{\theta=\alpha+1}^{\alpha+2}{}^{0}\mathbf{B}(\alpha,\theta)\sum_{k=0}^{\theta-\alpha-1}{}^{k}\boldsymbol{\psi}(k)\nu'(0,k)\\
&+\sum_{\theta=\alpha+1}^{\alpha+2}{}^{0}\mathbf{B}(\alpha,\theta)\,{}^{\theta-\alpha,\theta-\alpha}\mathbf{V}(\theta,\alpha+2)\nu(0,\theta-\alpha),
\end{aligned}
\tag{1.78}
$$

$$
\begin{aligned}
&{}^{0,0}\mathbf{V}(\alpha,\omega)=(\mathbf{I}-{}^{0}\mathbf{H}(\alpha,\omega))\sum_{k=0}^{\omega-\alpha-1}{}^{k}\boldsymbol{\psi}(k)\nu'(0,k)\\
&+\sum_{\theta=\alpha+1}^{\omega}{}^{0}\mathbf{B}(\alpha,\theta)\sum_{k=0}^{\theta-\alpha-1}{}^{k}\boldsymbol{\psi}(k)\nu'(0,k)\\
&+\sum_{\theta=\alpha+1}^{\omega}{}^{0}\mathbf{B}(\alpha,\theta)\,{}^{\theta-\alpha,\theta-\alpha}\mathbf{V}(\theta,\omega)\nu(0,\theta-\alpha).
\end{aligned}
\tag{1.79}
$$

1.6. The Dynamic Population Evolution Of The Pension Funds

We will now immediately consider an open pension scheme in a way which is defined below.
Let us begin with the following definitions:

${}^{\tau,h}N_i(s)$: the number of members present in the fund at time *h* with seniority τ, age *s* and in state *i,*
${}^{\tau,h}N(s)$: the number of members present in the fund at time *h* with seniority τ and age *s*,
${}^{\tau,h}N_i$: the number of members present in the fund at time *h* with seniority τ and in state *i,*
${}^{\tau,h}N$: the number of members present in the fund at time *h* with seniority τ ,

${}^{h}N_i$: the number of members present in the fund at time *h* in state i,
${}^{h}N$: the total number of members present in the fund at time *h*.

By summation, we clearly get.

$$ {}^{\tau,h}N = \sum_{s=\alpha}^{\omega} {}^{\tau,h}N(s) = \sum_{i=1}^{m-1} {}^{\tau,h}N_i \, , \tag{1.80} $$

$$ {}^{\tau,h}N_i = \sum_{s=\alpha}^{\omega} {}^{\tau,h}N_i(s), \ {}^{\tau,h}N(s) = \sum_{i=1}^{m-1} {}^{\tau,h}N_i(s) \, , \tag{1.81} $$

$$ {}^{h}N = \sum_{\tau=0}^{w} {}^{\tau,h}N = \sum_{i=1}^{m-1} {}^{h}N_i \, . \tag{1.82} $$

To go further, we have to introduce a scenario, often called the *central scenario*- also sometimes called the *basic strategy*- concerning the number of active people the firm wants to have at any time. In other words, this means that the value of

$$ {}^{h}N_i, h = 0,1,\dots,T \quad i = 1,2,\dots,m-5 \, , \tag{1.83} $$

where ${}^{h}N_i, h = 0,1,\dots,T$ represents the total number of people present in the fund at time *h* in state $i = 1,2,\dots,m-5$, *T* being a fixed *time horizon*.
From now on, we will work conditionally according to the observations resulting from this selected central scenario. In other words, this means that at time *h,* we know exactly the past counting evolution which may be used for computing probabilities of future events.
Now, we are to study the population fund on any time interval $\left[h, h+t-s\right)$, $(s < t)$; so let us define:

${}^{\tau+t-s,h}N(s,t)$: the number of members present in the fund initially of age *s* and seniority τ at time *h* still present at time $h+t-s$ $(t > s)$ and so of age *t*,
${}^{\tau+t-s,h}N_i(s,t)$: the number of members present in the fund initially of age *s*, and seniority τ at time *h* still present at time $h+t-s$ $(t > s)$ and so of age *t* but in state *i*.

With such counting observations, one could estimate some basic probabilities.
For example, let us consider the event that a member should definitively leave the system. More precisely, let ${}^{\tau+t-s,h}L(s,t)$ represent the probability that a member in state *i* with seniority τ at time *h* will go to the absorbing state *m* at time $h+t-s$.
Using the bi-dimensional associated semi-Markov process (1.29) with transition probabilities (1.30), we get:

$$ {}^{\tau+t-s}L(s,t) = \frac{\sum_{i=1}^{m-1} {}^{\tau}\phi_{im}(s,t)\, {}^{\tau,h}N_i(s)}{{}^{\tau,h}N(s)}. \tag{1.84} $$

It is now possible to compute the distribution of the counting variables ${}^{v+t-s,h}N_{(st)}$ in the following way:

$$ P\left({}^{\tau+t-s,h}N(s,t) = g \right) = \binom{{}^{\tau,h}N(s)}{g} \left(1 - {}^{\tau+t-s}L(s,t)\right)^g \left({}^{\tau+t-s}L(s,t) \right)^{{}^{\tau,h}N(s)-g}, \tag{1.85} $$

$g = 0,1,\ldots, {}^{\tau,h}N(s).$

More generally, using the multinomial distribution, we get for the joint distribution of the variables ${}^{\tau+t-s,h}N_i(s,t)$, i=1,...,m, the following result:

$$ \begin{aligned} & P\left({}^{\tau+t-s,h}N_1(s,t) = k_1, \ldots, {}^{\tau+t-s,h}N_m(s,t) = k_m \right) \\ & = \frac{\left(\sum_{i=1}^{m} {}^{\tau,h}N_i(s) \right)!}{k_1! \ldots k_m!} \left({}^{\tau+t-s,h}q_1(s,t) \right)^{k_1} \cdots \left({}^{\tau+t-s,h}q_m(s,t) \right)^{k_m}, \end{aligned} \tag{1.86} $$

where

$$ {}^{\tau+t-s,h}q_j(s,t) = \frac{\sum_{i=1}^{m} {}^{\tau}\phi_{ij}(s,t)\, {}^{\tau,h}N_i(s)}{{}^{\tau,h}N(s)}, j = 1,\ldots,m. \tag{1.87} $$

Let us recall that we are using a central scenario; also let us denote, at every time h, by Ψ_h all the information available including the past observations and also the scenario.

So, it is possible to compute conditional means at chosen times for some interesting counting variables. For example let us take the following one:

$$ {}^{\tau+t-s,h}\overline{N}_j(s,t) = E\left({}^{\tau+t-s,h}\overline{N}_j(s,t) \middle| \Psi_h \right). \tag{1.88} $$

From result (1.87), we get:

$$ {}^{\tau+t-s,h}\overline{N}_j(s,t) = {}^{\tau,h}N(s)\, {}^{\tau+t-s}q_j(s,t). \tag{1.89} $$

Another interesting conditional mean concerns the number of members being in state i, seniority τ and age s at time h and still not in the absorbing state at time $t - s + h$. If we note this conditional mean by ${}^{\tau,h}N_i(s,t)$, we immediately have:

$$ {}^{\tau,h}N_i(s,t) = \sum_{j=1}^{m-1} {}^{\tau}\phi_{ij}(s,t)\, {}^{\tau,h}N_i(s). \tag{1.90} $$

This mean represents at any time h the mean number of workers at time h+t−s we can predict at time h.

1.7 Financial Equilibrium Of The Pension Funds

Before studying the equilibrium of the fund, let us remark that the formulas we wrote before are useful for solving the equations in the general case but, if we want to face the problem of pension applications, there are some differences. In fact, as we said before, the age horizon of the fund member must be at least 85 years and the seniority horizon at most fifty years. If in writing formulas we were to be more precise we would take into account the maximum seniority. For this reason the formulas of evolution equations should be adapted.

Let us define:

K_s = maximum reachable seniority.

In this light the equations (1.43) and (1.44) can be written in the following way:

$$
{}^{\tau}\phi_{(st)} = \begin{cases} (I - {}^{\tau}H(s,t)) + \sum_{\theta=s+1}^{t} {}^{\tau}B(s,\theta)^{\tau+\theta-s}\phi(\theta,t); t \le K_s + s - \tau, \\ \\ (I - {}^{\tau}H(s,t)) + \sum_{\theta=s+1}^{K_s+s-\nu} {}^{\tau}B(s\theta)^{\tau+\theta-s}\phi(\theta,t) \\ + \sum_{\theta=K_s+1+s-\tau}^{t} {}^{\tau}B(s,\theta)^{K_s}P(\theta,t) \quad ;t > K_s + s - \tau. \end{cases} \tag{1.91}
$$

Also in this case there is no matrix inversion and the system can be solved in the same way as presented previously.

The presented model allows the management of a pension fund in which the considered rewards change because of the state, seniority and time and furthermore allows one to follow the *dynamic financial evolution* of the fund and therefore its *financial equilibrium*, the central objective of the fund managers.

$$
{}^{\tau,h}V_i(s,t)=\begin{cases}
(1-{}^{\tau}H_i(s,t))\displaystyle\sum_{k=h}^{h+t-s-1}{}^{\tau+k-h}\psi_{i(k+h)}\nu'(h,k)\\
+\displaystyle\sum_{\theta=s+1}^{t}\sum_{\beta=1}^{m}{}^{\tau}b_{i\beta(s\theta)}\sum_{k=h}^{h+\theta-s-1}{}^{\tau+k-h}\psi_i(k)\nu'(h,k)\\
+\displaystyle\sum_{\theta=s+1}^{t}\sum_{\beta=1}^{m}{}^{\tau}b_{i\beta}(s\theta)^{\tau+\theta-s,h+\theta-s}V_\beta(\theta,t)\nu(h,h+\theta-s),\\
t\le K_s+s-\tau,\\
(1-{}^{\tau}H_i(s,t))\displaystyle\sum_{k=h}^{h+K_s-\tau-1}{}^{\tau+k-h}\psi_i(k)\nu'(h,k)\\
+(1-{}^{\tau}H_i(s,t))\displaystyle\sum_{k=h+K_s-\tau}^{h+t-s-1}{}^{K_s}\psi_i(k)\nu'(h,k)\\
+\displaystyle\sum_{\theta=s+1}^{K_s+s-\tau}\sum_{\beta=1}^{m}{}^{\tau}b_{i\beta}(s,\theta)\sum_{k=h}^{h+\theta-\tau-1}{}^{\tau+k-h}\psi_i(k)\nu'(h,k)\\
+\displaystyle\sum_{\theta=K_s+s-\tau+1}^{t}\sum_{\beta=1}^{m}{}^{\tau}b_{i\beta}(s,\theta)\left(\sum_{k=h}^{h+K_s-\tau-1}{}^{\tau+k-h}\psi_i(k)\nu'(h,k)\right.\\
\left.\displaystyle\sum_{k=h+K_s-\tau}^{h+\theta-s-1}{}^{K_s}\psi_i(k)\nu'(h,k)\right)\\
+\displaystyle\sum_{\theta=s+1}^{K_s+s-\tau}\sum_{\beta=1}^{m}{}^{\tau}b_{i\beta}(s,\theta)^{\tau+\theta-s,h+\theta-s}V_\beta(\theta,t)\nu(h,h+\theta-s)\\
+\displaystyle\sum_{\theta=K_s+s-\tau+1}^{t}\sum_{\beta=1}^{m}{}^{\tau}b_{i\beta}(s,\theta)\Big({}^{\tau+\theta-s,h+\theta-s}V_\beta(\theta,t)\nu(h,h+\theta-s)\\
+{}^{K_s,h+\theta-s}V_\beta(\theta,t)\nu(h,h+\theta-s)\Big);\qquad t>K_s+s-\tau.
\end{cases}\tag{1.92}
$$

First of all let us define:
E: equity of the fund at time 0
A: total salary mean present value
O: total outlays mean present value
M: mean equilibrium rate (i.e. the percentage of the salary that is necessary for the equilibrium of the fund) defined as:

$$
M=\frac{O-E}{A}.\tag{1.93}
$$

To find the dynamic pension fund evolution, we need to know the M value and so it is necessary to make a static study of the fund.
Of course E is known and it suffices to compute O and A.
To obtain these values we need to solve twice the (1.92) and (1.93) evolution equations, the first time putting equal to 0 all the outlay rewards and equal to the

salary amount the positive (for the fund) rewards. We denote the reward vector obtained in this way $\boldsymbol{\psi}'$. The second time it is necessary to put equal to 0 all the positive rewards, giving the right values to the outlays. The related reward vector will be denoted $\boldsymbol{\psi}''$. These vectors are given in the following formulas:

$$ {}^{\tau}\boldsymbol{\psi}'(h)=\begin{bmatrix} {}^{\tau}\psi'_1(h) \\ \vdots \\ {}^{\tau}\psi'_{m-5}(h) \\ 0 \\ \vdots \\ 0 \end{bmatrix};h=0,1,\dots,w;\tau=0,\dots,K_s, \tag{1.94} $$

$$ {}^{\tau}\boldsymbol{\psi}''_k=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ {}^{\tau}\psi_{m-4}(h) \\ \vdots \\ {}^{\tau}\psi_m(h) \end{bmatrix};h=0,1,\dots,w;\tau=0,\dots,K_s. \tag{1.95} $$

Now we can solve twice the evolution equations (1.93) and we get respectively ${}^{\tau,h}V'_i(s,t)$, the *mean present value of the salaries* paid by the firm to a class of people that at time *h* had a seniority τ, an age *s* and was in the state *i* in the time period from *h* up to *h*+*t*−*s* and ${}^{\tau,h}V''_i(s,t)$, the *mean present value of the pensions* that the fund paid to a class of people that at time *h* had a seniority τ, an age *s* and was in the state *i* in the period from *h* up to *h*+*t*−*s*.
Now we can obtain the values of *A* and *O* as follows:

$$ A=\sum_{h=0}^{w}\sum_{t=\alpha}^{\omega}\sum_{s=\alpha}^{t}\sum_{\tau=0}^{\min(s-\alpha,K_s)}\sum_{i=1}^{m}{}^{\tau,h}V'_i(s,t)\,{}^{\tau,h}N_i(s,t), \tag{1.96} $$

$$ O=\sum_{h=0}^{w}\sum_{t=\alpha}^{K_a}\sum_{s=\alpha}^{t}\sum_{\tau=0}^{\min(s-\alpha,K_s)}\sum_{i=1}^{m}{}^{\tau,h}V''_i(s,t)\,{}^{\tau,h}N_i(s,t), \tag{1.97} $$

and finally the mean equilibrium rate *M* from relation (1.93).
By means of relations (1.96) (1.97) and (1.93) we get the *static study* of the fund in the sense of a knowledge of total outlay mean present value (technical reserve), of total salary mean present values and equilibrium rate.
Once we get the mean equilibrium rate *M*, we can follow the *dynamic development* of the pension fund.
Now let:

$$ {}^{\tau}\psi_j(h)={}^{\tau}\psi'_j(h)M,\quad j=1,\dots,m-5,\ \tau=0,\dots,K_s,h=0,\dots,w, \tag{1.98} $$

then, we have to solve (1.91) and we can obtain the ${}^{\tau+t-s,h}\overline{N}_j(s,t)$.

If I_k and O_k represent respectively the *total mean input* and *total mean output* for year k, we have:

$$I_k = \sum_{h=0}^{k}\sum_{s=\alpha}^{\omega}\sum_{\tau=0}^{K_s}\sum_{i=1}^{m-5} {}^{\nu,h}\overline{N}_i(s,s+k-h)\,{}^{\tau}\psi_i(h), \tag{1.99}$$

$$O_k = \sum_{h=0}^{k}\sum_{s=\alpha}^{\omega}\sum_{\tau=0}^{K_s}\sum_{i=m-4}^{m} {}^{\nu,h}\overline{N}_i(s,s+k-h)\,{}^{\tau}\psi_i(h). \tag{1.100}$$

By means of these last two results, we get respectively the annual entrances and outlays of the fund and we can follow its dynamic development.

1.8. Scenario And Data

The DTNHSM pension fund model presented here has been seen to be mathematically tractable, though it is a very general model, i.e., able to take into account a lot of possibilities such as age, seniority, salary line inflation rate,... and of course with the help of strong computer technology.
However, due to the long time horizon on which we study any pension fund, we must clearly distinguish between *scenario* and other relevant *data*.

A *scenario* is always represented by a set of data selected by the society or other authorities to study their influence in the future. There may be some changes in the rules of the pension scheme or in the future manpower planning of the society, thus such a scenario is never definitive and at any time it is possible to adapt it.
But there still remains the basic problem of the statistical estimation of the two-dimensional NHSM kernel given by (1.19). By (1.21), it suffices to estimate the ${}^{\tau}b_{ij}(s,t)$. To do so in a usual statistical way, we need not only a set of historical data big enough to accomodate classical statistical estimators but also relevant data concerning mortality experience tables, survivor distributions, etc.
Furthermore, we also need data concerning salary lines, seniority, inflation rate and so on.
Here too, let us mention that the model is very useful to project in the future some consequences related to changes in such data.
For example, we can suppose that the seniority of a new member always begins at 0, as we did above; we can also suppose some relation between the salary line and the inflation rate. Indeed it is possible that the increasing of salaries will just compensate the inflation rate modification or will just give 1% more.
To summarize, we may distinguish between three types of data sets:

(i) partial data given by a selected *internal scenario*,
(ii) historical data for estimation of the two-dimensional non-homogeneous semi-Markov kernel,

(iii) economic, financial and demographic data selected as an economic scenario predicting changes in a global economic environment.
This ability of the model to consider such possibilities is very important as it will give by simulation, in a completely rigorous way, the fund evolution on the selected time horizon.

Moreover, we can also proceed to quantitative comparisons for data as Khorasanee (1994) did.
These potentialities of the GDTNHSM pension fund model show that it constitutes a realistic approach to pension funding (see Thornton and Wilson (1993)).
Let us now give more details about the estimation of these data.

1.8.1 Internal Scenario

The first thing is to refer to the rules of the *pension scheme*. Of course some rules must be in keeping with the law and some other rules are special for the society under scrutiny.
At time 0, these are all known and this constitutes the given *central internal scenario,* but they may change in the sequel and some changes may be included and then constitute another internal scenario.
As said above, this type of scenario can also include future *manpower planning.* This means that it depends on *strategic decisions* of the society for its future development and not only for survival of the pension fund. This strategy will give the future counting values for all active states.
The total choice at time h constitutes the information Ψ_h given in section 1.6.

Remark 1.1 More theoretically the sequence $\left(\Psi_h, h \geq 0\right)$ constitutes a filtration that must be added to a basic probability space $\left(\Omega, \Im, \Re\right)$ but for simplicity we will not develop these more theoretical aspects here.

1.8.2 Historical Data

The general methodology for estimating the ${}^{\tau}b_{ij}(s,t)$ is to go from s to $s+t$ with successive unit steps so that we can find the estimation values in related statistical tables.
For example, let us consider the element ${}^{\tau}b_{ai}(s,t)$ where a is one of the $m-5$ active states and i the disability state.

Figure 1.8: times of statistical estimation

By means of classical statistical methods,, functions ${}^{\tau}F_{ij}(s,t)$ and ${}^{\tau}p_{ij}(s)$ can be estimated. Taking into account relation (1.27) it is then possible to estimate the ${}^{\tau}Q_{ij}(s,t)$ and at last we obtain:

$$ {}^{\tau}b_{ij}(s,t)=\begin{cases} 0 & s\geq t, \\ {}^{\tau}Q_{ij}(s,t)-{}^{\tau}Q_{ij}(s,t-1) & s<t. \end{cases} \tag{1.101} $$

Of course analog formulas may be easily found for the other possible transitions. Usually the necessary probabilities are available from actuarial tables.
Special attention is necessary to mortality tables as it is now well known that, at the present time, for most Western countries the mean lifetime increases by three months every four years! It is clear that this fact already explains the great anxiety of pension fund managers concerning the financial equilibrium of their funds.
Once more, this fact shows that the management of pension funds must be continuous to take into account new reliable information.

1.8.3 Economic Scenario

The simplest assumption, often made in the study of pension funds, is that the real interest rate is *constant* over all time horizon. However, we prefer to create scenarios both for the inflation rate and the market rate on the time horizon.
This is really the most difficult problem, but we can always use Fisher's relation to find the real interest rate.
It is also possible to find help in e.g. the Wilkie model (Wilkie (1994)) and in recent evolution in interest rate models in stochastic finance.
Anyway we have to choose and this choice is a factor in our economic scenario.
For the salaries we propose to adopt the following strategy: let us construct a three-way matrix where one index represents the *hierarchical rank* and the other two respectively *time* and *seniority*.
We emphasize that the model includes the change in hierarchical ranks with the transitions between states 1 to $m-5$. So we only need to construct the salary line for each rank, which can be seen as a sheet of our three-way salary matrix.
Clearly, salary lines for each rank at time 0 are known. This gives the following matrix:

$$ \mathbf{M}_i=\left[m_i(\tau,t)\right] \quad i=1,...,m-5,\ \tau=1,...,K_s,\ t=1,...,w \tag{1.102} $$

where $m_i(\tau,h)$ represents the salary rank in state *i*, (i=1,...,$m-5$) and seniority τ at time *h*. This spreadsheet can be used as a Lexis diagram for each active member of the fund.
Now we have to adapt an index evolution of the salary to go from row 0 to the following ones, and so to know the general elements of matrix $\mathbf{M}_i$.
An important part of an economic scenario is to select indexes $r'_0, r'_1, ..., r'_h, ...$ representing the rates at which we decide the salary will increase.
So we have the following relations:

$$m_i(\tau,h+1) = (1+r'_h)m_i(\tau,h) \quad (1.103)$$

solving the problem of *salary line evolution*.
The same problem remains for the *evolution of pension amount* but fortunately in a simple way.
In fact we can suppose that all the pensions grow with only one index rate. This means that this evolution is independent of the obtained hierarchical rank of the member just before getting a pension, but it is time dependent. So we need just to select a vector $r''_0, r''_1, ..., r''_h, ...$ as another part of our economic scenario.
Let us finally remark that this last choice is also very important from the political point of view, particularly for relations between governments, trade unions and management representatives.

Remark 1.2.
(i) All the parameters we introduce in the economic scenario will in fact define s*olidarity* inside the fund.
Also, large simulations can give a financial measure of matching increases of lower pensions and decreases or stability of higher pensions.
(ii) The evolution of disability and survivor pensions can be treated in the same way.

1.9 Usefulness Of The NHSMPFM

(i) As we said in the introduction, to our knowledge, the GDTNHSM pension fund model presented here gives for the first time a general, rigorous and tractable framework for studying the time evolution of a fund, taking into account economic, financial and demographic possibilities of changes.
(ii) Its use as a *simulation model* will provide a very powerful internal tool, for society or for legal authorities, to measure the influences of these modifications.
(iii) The presentation of our GDTNHSM model given here is *microeconomic*, i.e. related to one society, but its extension as a *macroeconomic* model to be used by regional or national authorities or also big insurance companies is straightforward. The only new problem is that of how to aggregate the data

necessary to start with the model and it may be judicious not to consider too many states.
(iv) Let us also mention that, at least in our opinion, in the future more and more members of pension funds will take out private pensions contracted with insurance companies. The GDTNHSM model can be useful to fix the amount of premium to be paid for this supplementary pension in connection with the "usual" pension amount.
(v) From the computational point of view, it is clear that this model cannot be used as a simulation model without a good computer environment giving an easy way to measure the influence of the selected scenario. Interactive software is now being prepared by the authors in collaboration with some private pension funds.

2. GENERALIZED NON-HOMOGENEOUS SEMI-MARKOV MODEL FOR MANPOWER MANAGEMENT

2.1 Introduction

In the preceding section, we have seen that the evolution of salary lines is one of the most important aspects in the study of the dynamic evolution of pension funds, so we need to construct a model giving a good forecast of future salary lines. Of course, this model can also be used for other aims, in manpower management for example when a firm decides to change the rules of the rank promotions or its pyramidal job organisation. In this case, it is interesting to evaluate the cost differences between the new and the former rules in view of improving the manpower planning of the firm (Bartholomew (1982), Bartholomew, Forbes, McClean (1991), Vajda (1978).)
In this section, we solve the problem by giving a *generalisation* of non-homogeneous semi-Markov processes slightly different from the one used in the previous section.
This problem was treated by Volpe (1997), Janssen and Manca (1997b) (2002) Janssen, Manca, Volpe (1997) Manca (2004a) (2004b), using different kinds of stochastic models.
Let us remember that the evolution of salary lines has a strong influence on the behaviour of pension funds.
Indeed, if the fund is *with defined contribution* (that means the pension is a function of the paid contributions) then it is important to know the expected evolution of the salary lines to know what the entrances into the fund will be for each member and in this way to compute the expected pensions for the working people in the fund.
Furthermore if the fund is *with defined benefit (performance)* (that means the pension is a function of the last salaries), it is important to know the evolution of

salary from the point of view of fund entrances, but in this case the pension is directly a function of the last salaries paid to the members, so it is possible to understand the relevance of salary lines in this case.
Here, we face the problem of making a generalisation of a model useful to forecast the development of pension funds (Janssen and Manca (1997a)), presented in the preceding section, which is an extension of the discrete time non-homogeneous semi-Markov processes (see for example Janssen and De Dominicis, (1984)) presented in Chapter 4.
Markov processes and semi-Markov processes were already used in the man-power planning problems, see for example Bartholomew (1982).

2.2 GDTNHSMP For The Evolution of Salary Lines

A model for salary line evolution uses a state space with $m-1$ possible ranks in the active state and an absorbing state *m* representing the state of leaving the job for retirement, for death or for any other reason.
As usual, let us denote this state space by *I* with

$$I=\{1,...,m\}. \tag{2.1}$$

Let us introduce now a discrete time scale: we observe the state at times 0,1,2,...,*k*,....and so, clearly, at any time *k*, each member of the company is in one and only one of these *m* states.
A quadruple of random variables (J_n,T_n,H_n,K_n) is now introduced, where J_n represents the state of the member at transition *n*, T_n the time in which there is the transition *n*, S_n the seniority of the member at transition n and K_n the age of the member at transition *n*.

Remark 2.1 Here seniority means the effective number of years that the member is in the company.

The seniority S_{n+1} and the age K_{n+1} are usually defined by the relations:

$$S_{n+1}=S_n+T_{n+1}-T_n, \tag{2.2}$$

$$K_{n+1}=K_n+T_{n+1}-T_n. \tag{2.3}$$

Now, at each state transition, the considered member of the company is characterised by the quadruple

$$(J_n,T_n,S_n,K_n). \tag{2.4}$$

This (*J,T,S,K*) process may in fact also be considered as a three-dimensional non-homogeneous Markov renewal process (*J*,(*T*-*S*),(*T*-*K*)) with kernel

$${}^{\mu,\tau}Q_{ij}(s,t)=P\begin{pmatrix} J_{n+1}=j,T_{n+1}\le t,S_{n+1}\le\tau+t-s,K_{n+1}\le\mu+t-s \\ |J_n=i,T_n=s,S_n=\tau,K_n=\mu \end{pmatrix}. \tag{2.5}$$

Now, we define the matrix **B** having as general element:

$$ {}^{\mu,\tau}b_{ij}(s,t) = P\begin{pmatrix} J_{n+1} = j, T_{n+1} = t, S_{n+1} = \tau + t - s, K_{n+1} = \mu + t - s \\ \big| J_n = i, T_n = s, S_n = \tau, K_n = \mu \end{pmatrix} \tag{2.6} $$

so that:

$$ {}^{\mu,\tau}b_{ij}(s,t) = \begin{cases} 0, & s \geq t, \\ {}^{\mu,\tau}Q_{ij}(s,t) - {}^{\mu,\tau}Q_{ij}(s,t-1) & s < t, \end{cases} \tag{2.7} $$

or equivalently:

$$ {}^{\mu,\tau}Q_{ij}(s,t) = \sum_{h=s}^{t} {}^{\mu,\tau}b_{ij}(s,h). \tag{2.8} $$

Similarly, it is possible to write:

$$ {}^{\mu,\tau}p_{ij}(s) = P\left(J_{n+1} = j \,\middle|\, J_n = i, T_n = s, S_n = \tau, K_n = \mu\right), \tag{2.9} $$

representing the general element of the kernel of the process.
We also have:

$$ {}^{\mu,\tau}p_{ij}(s) = {}^{\mu,\tau}Q_{ij}(s,\infty), \tag{2.10} $$

$$ {}^{\mu,\tau}H_i(s,t) = \sum_{j=1}^{m} {}^{\mu,\tau}Q_{ij}(s,t), \tag{2.11} $$

where:

$$ {}^{\mu,\tau}H_i(s,t) = P\left(T_{n+1} \leq t \,\middle|\, J_n = i, T_n = s, S_n = \tau, K_n = \mu\right), \tag{2.12} $$

$$ {}^{\mu,\tau}H_i(s,\infty) = 1, \tag{2.13} $$

$$ {}^{\mu,\tau}F_{ij}(s,t) = \frac{{}^{\mu,\tau}Q_{ij}(s,t)}{{}^{\mu,\tau}p_{ij}(s)}, \tag{2.14} $$

where

$$ {}^{\mu,\tau}F_{ij}(s,t) = P\left(T_{n+1} \leq t \,\middle|\, J_n = i, J_{n+1} = j, T_n = s, S_n = \tau, K_n = \mu\right), \tag{2.15} $$

i.e. the sojourn time conditional distribution entering in state i at transition n with a seniority τ at age μ.
Finally, it is also possible to relate the associated semi-Markov process in keeping with the three-dimensional non-homogeneous Markov renewal process $(J,(T\text{-}S),(T\text{-}K))$ noted as

$$ \left({}^{\mu,\tau}Z_t; t \geq 0, \tau \geq 0, \mu \geq 0\right) \tag{2.16} $$

having as transition probabilities:

$$ {}^{\mu,\tau}\phi_{ij}(s,t) = P\left({}^{\mu+t-s,\tau+t-s}Z_t = j \,\middle|\, {}^{\mu,\tau}Z_s = i\right). \tag{2.17} $$

These transition probabilities satisfy the following system:

$$ {}^{\mu,\tau}\phi_{ij}(s,t) = \delta_{ij}(1 - {}^{\mu,\tau}H_i(s,t)) + \sum_{\theta=s}^{t}\sum_{h=1}^{m} {}^{\mu,\tau}b_{ih}(s,\theta)\, {}^{\mu+\theta-s,\tau+\theta-s}\phi_{hj}(\theta,t). \tag{2.18} $$

It should be emphasized that the probabilities (2.9) are related to the generalized non-homogeneous Markov process embedded in the generalized non-homogeneous semi-Markov process (2.16).
Furthermore ${}^{\mu,\tau}\phi_{ij}(s,t)$ represents the probability of staying in state j at time t once the process was in state i at time s with a seniority τ at age μ.

2.3 The GDTNHSMRWP For Reserve Structure

To apply this model to the computation of present value of the current salary cost, it is necessary to consider a reward structure connected to the semi-Markov process.
Clearly the salary value changes because of time and seniority but it does not change as a function of age, however the probability of being promoted and the transition to the absorbing state can be related to age.
For this reason it is necessary to write the reward equation taking into account all these aspects. The equation will be slightly different from relation (1.39) because time is the main temporal variable.
In this light, relation (1.39) becomes:

$$
\begin{aligned}
{}^{\mu,\tau}V_i(s,t) &= (1-{}^{\mu,\tau}H_i(s,t))\left(\sum_{\theta=s}^{t} {}^{\tau+\theta-s}\psi_i(\theta)\left(1+r\right)^{s-\theta}\right) \\
&+\sum_{\theta=s}^{t}\sum_{\beta=1}^{m} {}^{\mu,\tau}b_{i\beta}(s,\theta)\sum_{\vartheta=0}^{\theta-1} {}^{\tau+\vartheta}\psi_i(s+\vartheta)\left(1+r\right)^{-\vartheta} \\
&+\sum_{\theta=s}^{t}\sum_{\beta=1}^{m} {}^{\mu,\tau}b_{i\beta}(s,\theta)\, {}^{\mu+\theta-s,\tau+\theta-s}V_\beta(\theta,t)(1+r)^{s-\theta}.
\end{aligned}
\tag{2.19}
$$

In the case of the salary cost evaluation model, the financial meaning of the equations (2.19) can be given.
The ${}^{\mu,\tau}V_i(s,t)$ are the discounted expected values of the salaries that were paid from s to t when an employee was in rank i at time s with a seniority τ and an age μ. These formulas are compounded from the following parts:

$$(1-{}^{\mu,\tau}H_i(s,t))\left(\sum_{\theta=s}^{t} {}^{\tau+\theta-s}\psi_i(\theta)\left(1+r\right)^{s-\theta}\right), \tag{2.20}$$

$$\sum_{\theta=s}^{t}\sum_{\beta=1}^{m} {}^{\mu,\tau}b_{i\beta}(s,\theta)\sum_{\vartheta=0}^{\theta-1} {}^{\tau+\vartheta}\psi_i(s+\vartheta)\left(1+r\right)^{-\vartheta}, \tag{2.21}$$

$$\sum_{\theta=s}^{t}\sum_{\beta=1}^{m} {}^{\mu,\tau}b_{i\beta}(s,\theta)\, {}^{\mu+\theta-s,\tau+\theta-s}V_\beta(\theta,t)(1+r)^{s-\theta}. \tag{2.22}$$

In relation (2.20), the term $(1-{}^{\mu,\tau}H_i(s,t))$ represents the probability of remaining in state i once a member has arrived at time s with a seniority τ and

an age μ. So this member for each time period gets the salary ${}^{\tau+\theta-s}\psi_i(\theta)$, depending on the time and seniority and so (2.20) represents the expected related value.
Relation (2.21) gives the expected present value of the salaries that a member arrived at i at time s with seniority τ and age μ got in this state before it changed.
Finally, the expression (2.22) represents the expected value of the salaries that a member, having arrived in state i at time s with a seniority τ and an age μ and having changed his situation at time θ, has to get in the new state.
These values are paid at time $\theta - s$, so it is necessary to discount them.
In this way, the probabilities of changing states differ because of seniority, but it is also possible to consider different rewards as a function of different seniorities.

2.4 Reserve Structure With Stochastic Interest Rate

A stochastic interest rate structure is introduced in this part.
The structure will be constructed by means of DTNHSMP as in Chapter 6.
In this case the state of the process will be:

$$E = \{r_1, r_2, \ldots, r_k\} \tag{2.23}$$

where the r_i represents all the possible implied stochastic interest rates and k gives the number of the implied interest rates.
Now $\phi_{ij}(s,t)$ represents the probability that at time t the implied interest rate will be r_j, given that the implied interest rate was r_i at time s and $\nu_i(s,h)$ represents the mean discounting factor for a time $(h-s)$ given that at time s the interest rate was r_i.
Now, the evolution equation (2.19) becomes:

$$\begin{aligned}
{}^{\mu,\tau}V_i^{\varepsilon}(s,t) &= (1 - {}^{\mu,\tau}H_i(s,t))\left(\sum_{\theta=s}^{t} {}^{\tau+\theta-s}\psi_i(\theta)\nu_{\varepsilon}(s,\theta)\right) \\
&+ \sum_{\theta=s}^{t}\sum_{\beta=1}^{m} {}^{\mu,\tau}b_{i\beta}(s,\theta)\sum_{\vartheta=0}^{\theta-1} {}^{\tau+\vartheta}\psi_i(s+\vartheta)\nu_{\varepsilon}(s,s+\vartheta) \\
&+ \sum_{\theta=s}^{t}\sum_{\beta=1}^{m} {}^{\mu,\tau}b_{i\beta}(s,\theta)\, {}^{\mu+\theta-s,\tau+\theta-s}\overline{V}_{\beta}^{\varepsilon}(\theta,t)\nu_{\varepsilon}(s,\theta),
\end{aligned} \tag{2.24}$$

assuming that r_{ε} will be the known interest rate at time s.
From (7.8) of Chapter 10 it results that:

$${}^{\mu,\tau}\overline{V}_{\beta}^{\varepsilon}(\theta,t) = \sum_{j=1}^{n} \phi_{\varepsilon j}(s,\theta)\, {}^{\mu,\tau}V_{\beta}^{j}(\theta,t) \tag{2.25}$$

so that we obtain the following results:

$$^{\mu,\nu}V_i(s,t) = \sum_{\eta=1}^{k} {}^{\mu,\nu}V_i^{\eta}(s,t)\phi_{\eta\varepsilon}(0,s) \tag{2.26}$$

where r_η is the known rate of interest at time 0.

The solution of the GDTNHSMP and the GDTNHSMRWP can be obtained by a backward substitution process similar to the one described in section 1 for the pension model.

2.5 The Dynamics Of Population Evolution

Also the population evolution can be studied by means of the same relations given in section 1.6.
By means of relations given in that part, it is possible to evaluate the number of people in the ranks of a given company and furthermore taking into account the possibility of new additions to the workforce.
We report only the definitions because of the different uses of temporal variables as the related formulas can be obtained in the same way as given in the previous part.

$^{\mu,\tau}N_i(s)$: the number of members present in the company in rank i at time s with seniority τ and age μ,
$^{\mu,\tau}N(s)$: the number of members present in the company at time s with seniority τ and age μ,
$^{\tau}N_i(s)$: the number of members present in the company in rank i at time s with seniority τ,
$^{\tau}N(s)$: the number of members present in the company at time s with seniority τ,
$N_i(s)$: the number of members present in the company in rank i at time s,
$N(s)$: the total number of members present in the company at time s.

Here too, we can introduce a *scenario* concerning the number of active people the firm wants to have at any time in each rank; the values at time 0 being known:

$$N_i(s), s = 0,1,\dots,T \;;\;\; i = 1,2,\dots,m-1\,, \tag{2.27}$$

where, as usual, T is the fixed time horizon.
All the relations given previously hold and will not be repeated.
Taking into account the different meanings of the temporal variable and the given scenario, we can evaluate the $^{\mu,\tau}N_i(s)$ values by means of the relations given in section 1.6. These values represent the mean number of people present at time s with seniority τ and age μ in state i.

Then we get the mean charge of future work at any time t that can be predicted at present time s ($s<t$):

$$ {}^{\mu,\tau}N_i(s,t) = \sum_{j=1}^{m-1} {}^{\mu,\tau}\phi_{ij}(s,t)\,{}^{\mu,\tau}N_i(s). \tag{2.28}$$

The seniority and the age are also known and it is clear that all these results are fundamental for the future labour policy of the firm.

2.6 The Computation Of Salary Cost Present Value

Once the systems (2.18) and (2.19) are solved, given a person that is at time s with seniority τ and age μ in state i, the probabilities:

$$ {}^{\mu,\tau}\phi_{ij}(s,s+1),\ {}^{\mu,\tau}\phi_{ij}(s,s+2),\ {}^{\mu,\tau}\phi_{ij}(s,s+3),\cdots,\ {}^{\mu,\tau}\phi_{ij}(s,w);\ j=1,\ldots,m-1 \tag{2.29}$$

are known.
They represent the probabilities of being in rank j after one year, two years and so on; these probabilities will be equal to 0 if they represent some impossible cases.
By means of results (2.28) and (2.29), we can obtain all the ${}^{\mu,\tau}N_i(s,t)$ that are necessary.
Furthermore for each year it is possible to compute the salary line, if the salary that will be paid to a person with a given seniority in a given rank for each year is known, which in general is the case.
If we suppose that the rates of salary evolution are known for the given company (we suppose we are in a given *economic scenario*), then it is possible to construct, for each rank, a matrix the elements of which will give the expected salary in the firm for each seniority and each year in our time horizon.
In this way we will obtain a three-dimensional matrix with as elements:

$$ {}^{\tau}\psi_j(s);\quad j=1,\ldots,m-1, \tau=0,\ldots,K_s,\ s=0,\ldots,w \tag{2.30}$$

representing the salary in state j, of a person with seniority τ at time s in the given *scenario*.
The mean salary line of a given situation in the time horizon, as specified in Janssen Manca Volpe (1997), will be given by:

$$ {}^{\mu,\tau}A_i(s,t) = \begin{cases} \sum_{j=1}^{m-1} {}^{\mu,\tau}\phi_{ij}(s,t)\,{}^{\tau+t-s}\psi_j(t); & \tau+t-s \le K_s \\ \sum_{j=1}^{m-1} {}^{\mu,\tau}\phi_{ij}(s,t)\,{}^{K_s}\psi_j(t); & \tau+t-s > K_s \end{cases} ;\ t=s+1,\ldots,w. \tag{2.31}$$

To apply the model it is necessary to know all the ${}^{\mu,\tau}b_{i,j}(s,t)$. To obtain these probabilities it is necessary to know all the functions ${}^{\mu,\tau}Q_{i,j}(s,t)$, obtained, taking into account relation (2.12), knowing ${}^{\mu,\tau}F_{i,j}(s,t)$ and ${}^{\mu,\tau}p_{i,j}(s)$.

Though not easy to be found, these data may be obtained by observation and there are some models that can be used (see for example De Dominicis and Manca (1984a)).
Once the data are known, it is possible to solve (2.18) and (2.19) and to obtain the results.
Finally we get the following formula:

$$A=\sum_{\tau=0}^{K_s}\sum_{\mu=\alpha}^{K_{mwa}}\sum_{i=1}^{m} {}^{\mu,\tau}V_i(0,w)\,{}^{\mu,\tau}N_i(0,w)$$
$$+\sum_{\mu=\alpha}^{K_{mwa}}\sum_{s=1}^{w}\sum_{i=1}^{m} {}^{\mu,0}V_i(s,w)\,{}^{\mu,0}N_i(s,w)(1+r)^{-s} \quad (2.32)$$

giving the present value of salary cost, where the first part represents the cost for the people that are in the company at time 0 and the second part the salary cost of the people that will enter into the company.
As already specified, the number of the new employees is a function of the scenario in which the model develops.
Finally, let us remark that the salary line forecasting problem can be solved also by means of other models based on the generalizations of the Bernoulli stochastic process (Manca (2004a)).
However in this case the probabilities ${}^{\mu,\tau}\phi_{ij}(s,s+1)$ become inputs for the generalized binomial process.

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